Complex Stock Price Dynamics and Recurrent Bubbles under the Spirit of Capitalism

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Complex Stock Price Dynamics and Recurrent Bubbles under the Spirit of Capitalism*

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Abstract

We embed Max Weber (1958)'s spirit of capitalism (SOC) into an otherwise standard Lucas’ tree asset pricing model, by assuming that economic agents care about their social status and that the latter is related to financial wealth. We show that, absent uncertainty, for a wide range of values for the coefficient of relative risk aversion, the SOC combined with a sufficient growth rate of dividends can induce both cyclical and chaotic fluctuations in the price-dividend ratio. Once fundamental uncertainty is introduced - in the form of a stationary shock to dividends growth - the model is capable of generating recurrent boom-bust cycles which are completely endogenous and resemble the stock price bubble episodes observed in the real world. Interestingly, these volatile dynamics are obtained without necessarily assuming the existence of sizable (sunspot-driven) expectational errors.

We show that, for both the deterministic and the stochastic case, endogeneous fluctuations can be effectively smoothed by a small tax on capital gains, while taxing dividends seems to be a rather bad idea.

Keywords: Asset Pricing, Bubbles, Endogenous Cycles, Chaotic Dynamics, Status, Wealth Preferences, Spirit of Capitalism, Dividends Taxation, Capital Gains Taxation

JEL Classifications: C61, C62, E21, E32, E62, G12

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1 Introduction

It is a well known fact that stock prices are much more volatile than the underlying fundamentals. The US price to dividend ratio, for instance, in the post-war period has fluctuated between the values of 20 and 30, to suddenly jump to almost 100 in the mid-90s and then revert back to more regular levels after 2000. Meanwhile, the underlying fundamentals - namely real dividends and earning - remained fairly stable.

While many economists have often attributed these sharp asset price booms to investors’ irrationality, the recent financial turmoils have brought back the idea that asset price bubbles can result from perfectly rational decision-making by fully informed economic agents, an idea originally developed by Tirole (1985). Though fully cognizant about the underlying fundamental value, agents might be willing to pay more for an asset if they expect its price to further increase in the future. Despite their theoretical consistency and plausibility, under this paradigm, stock price end up growing faster than dividends, displaying an ever-growing path that is clearly at odds with the data.

To correct for this unrealistic feature, more sophisticated models have introduced \textit{ad-hoc} periodical crashes, which occur with a probability well-known to everyone. While this addiction clearly helps improving the empirical fit, it leaves outside the model why crashes occur and, more importantly, it does not allow for subsequent bubble episodes. Lansing (2010) makes an interesting contribution in this respect, showing that we can build near-rational bubble solutions that preserve volatility without implying ever growing equilibrium paths. In his paper, bubbles are intrinsic - i.e. they are related to fundamentals, and hence preserve stationarity - and recurrent - i.e. they expand and contract in a completely endogenous way. A similar pattern is obtained by Adam, Marcet and Nicolini (2008) through the introduction of boundedly-rational agents who need to (adaptively) learn from the past to make forecasts about future variables.

In this paper, we take a different route. Rather than further exploring the quantitative implications of alternative forms of expectations formation and learning, we maintain the rational expectations hypothesis in place and focus on agents’ preferences. We consider a standard Lucas’ tree asset pricing model and introduce Max Weber (1958) spirit of capitalism hypothesis. More specifically, we assume that economic agents care about their own social status, which we assume being closely related to financial wealth, similarly to Bakshi and Chen (1996), Smith (2001) and Kamihigashi (2007). Within this framework we study the possibility of endogenous non-fundamental stock price fluctuations due to consumption and status entering non-separably in people preferences.

We show that, absent uncertainty, for a wide range of values for the coefficient of relative risk aversion, the SOC combined with a sufficient growth rate of dividends can induce both cyclical and chaotic fluctuations in the price-dividend ratio. If chaotic, stock prices display fluctuations that appear erratic and irregular, but non-explosive. Exploding bubbles are possible - and consistent with optimality - at lower level of risk aversion. In this sense, we nest the results of Kamihigashi (2007), who shows that under the spirit of
capitalism and separable preferences stock price bubbles are viable equilibria, since along those path all optimality conditions - and in particular the transversality condition - perfectly hold.

Once fundamental uncertainty is introduced - in the form of a shock to dividends growth - the model is capable of generating recurrent boom-bust cycles which are completely endogenous and resemble the stock price bubble episodes observed in the real world. Interestingly, these volatile dynamics are obtained without necessarily assuming the existence of sizable (sunspot-driven) expectational errors. We assess how the existence of these boom-bust phases depends on the volatility of the fundamental shock, its persistence, as well as on the long-run mean of the dividends growth rate. While for the deterministic case, cycles and chaos require a moderate degree of dividends growth, under uncertainty, recurrent bubbles and crashes can occur at rather small growth rates.

With complex dynamics arising because of structural reasons, one might wonder whether there exist appropriate policies to eliminate the excessive stock price volatility. In this respect, we explore the consequences of taxing either dividends or capital gains. Interestingly, we find that a minimal tax on capital gains can eliminate both cycles and chaos, while dividends taxes have the opposite effects. Other things equal, an increase in the dividends tax rates lowers the minimum growth rate above which period-2 cycles occur. The stabilizing (respectively, destabilizing) properties of capital gains (respectively, dividends) taxation also hold for the stochastic version of our economy.

The paper is structured as follows. Section 2 develops the model and defines the relevant equilibrium concepts. Section 3 characterizes the steady state, the existence of cyclical and chaotic dynamics, and provides some quantitative evaluation of the theoretical results. Section 4 studies the case of both sunspot and fundamental shocks, showing the existence of boom-busts phases via numerical simulations. Section 5 discusses about the stabilizing properties of dividends and capital gains taxes. Section 6 extends the model to allow for aggregate wealth in status and for endogenous dividends. Section 7 concludes.

2 The model

The economy is populated by a continuum of identical infinitively-lived agents. The representative agent seeks to maximize his expected intertemporal utility:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, s_t)$$

where the instantaneous utility $U(c_t, s_t)$ is a function of consumption $c_t$ and status $s_t$. The utility function $U(\cdot)$ is strictly increasing and concave in both arguments and satisfies standard conditions. As in Bakshi and Chen (1996), Smith (2001) and Kamihigashi (2007), we incorporate Max Weber’s "spirit of capitalism" (SOC) in our economy by assuming that status is related to wealth. More specifically, we assume that $s_t$ is
defined by:

\[ s_t \equiv \frac{w_t}{(W_t)^\mu} \text{ for } \mu \in [0, 1] \tag{1} \]

where \( w_t \) is the agent’s own financial wealth, \( W_t \) is aggregate financial wealth and the coefficient \( \mu \) captures how important is aggregate wealth for the individual status. This specification nests the case where status is related to own wealth only (\( \mu = 0 \)), as well as the case where status depends on relative wealth (\( \mu = 1 \)). But more generally, the higher is \( \mu \) the more the agent’s status is negatively affected by other people’s wealth.\(^1\)

In order to convey the main results, we focus on the following specific functional forms:

\[
U(c_t, s_t) = \frac{X_t^{1-\sigma}}{1-\sigma} \tag{2}
\]

\[
X_t = \left[ (1-\alpha) c_t^{\xi-1} + \alpha s_t^{\xi-1} \right]^{\frac{1}{\xi-1}} \tag{3}
\]

where \( \sigma > 0, \xi > 0 \) and \( \alpha \in (0,1) \). The coefficient \( \alpha \) captures the relative importance of status versus consumption in the household’s preferences, while \( \xi \) is the elasticity of substitution between consumption and wealth in the aggregator \( X_t \).

There is only one asset in this economy, which we call "stock". In every period \( t \), the stock generates dividends \( d_t \) in an exogenous fashion like the Lucas’ tree model. Shares of the stock are sold to the agents populating the economy at market prices. The representative agent’s budget constraint is the following:

\[
c_t + p_t a_t = (p_t + d_t) a_{t-1}
\]

where \( a_t \) is the number of shares of the stock purchased at time \( t \), at the per unit price \( p_t \). At the beginning of each period \( t \) the agent receives dividends payments of \( d_t \) per share purchased at \( t-1 \), while his initial equity portfolio is \( p_t a_{t-1} \). Individual financial wealth is therefore \( w_t = p_t a_t \). Notice that for \( \alpha = 1 \) the model collapses to a standard Lucas’ tree asset pricing model with CRRA utility.

The Euler equation and the transversality conditions associated with this optimization problem are:

\[
\beta U_c (c_{t+1}, s_{t+1}) (p_{t+1} + d_{t+1}) = p_t U_c (c_t, s_t) \left[ 1 - \frac{U_s (c_t, s_t)}{U_c (c_t, s_t) W_t^\mu} \right] \tag{4}
\]

\[
\lim_{t \to \infty} \beta^t [U_c (c_t, s_t) - U_s (c_t, s_t)] p_t a_t = 0 \tag{5}
\]

To start with, we are going to consider the case of a unit elasticity of substitution between consumption and status (\( \xi \to 1 \)) - which corresponds to the case of a Cobb-Douglas aggregator \( X_t \) - and assume that

\(^1\)In a similar context, one could consider the case where status depends on real estate wealth. This would probably be more related to the global financial crisis started at the end of 2007.
status is equal to individual wealth \( \mu = 0 \). Later on, we will relax both assumption and consider a more general set-up.

**Assumption 1:** \( \xi = 1 \)

**Assumption 2:** \( \mu = 0 \).

Under these Assumptions 1 and 2, the instantaneous utility becomes \( U(c_t, s_t) = (c_t^{1-\sigma}w_t^{1-\sigma}) \). After imposing the equilibrium conditions \( c_t = d_t \) and (without loss of generalitt) \( a_t = 1 \), and hence \( w_t = p_t \), from (4), we obtain:

\[
\beta d_t^{(1-\alpha)(1-\sigma) - 1} p_t^{(1-\sigma)\alpha} (p_{t+1} + d_{t+1}) = d_t^{(1-\alpha)(1-\sigma) - 1} p_t^{(1-\sigma)\alpha} \left( 1 - \frac{\alpha}{1 - \alpha p_t} \right) d_t
\]

(6)

Since we will focus on the possibility of endogenous fluctuations, we assume a completely deterministic path for dividends. More specifically, we assume that dividends grow at a constant gross rate \( \gamma \) per period.²

**Assumption 3:** \( d_{t+1} = \gamma d_t \), with \( \gamma \geq 1 \) and \( d_0 = d > 0 \).

Letting \( q_t \equiv \frac{d_t}{p_t} \) be the dividend-price ratio, we can write the dynamic equation (6) in a more compact form:

\[
q_{t+1}^{\chi} (1 + q_{t+1}) = \frac{\gamma^{(\sigma-1)}}{\beta} q_t^\chi \left( 1 - \frac{q_t}{\phi} \right)
\]

(7)

where:

\[
\chi \equiv (\sigma - 1) \alpha - 1
\]

(8)

\[
\phi \equiv \frac{1 - \alpha}{\alpha} > 0
\]

(9)

From (7), notice that along any equilibrium path we must have \( q_t \in (0, \phi) \). Clearly, \( q_t < 0 \) is not feasible, as it would imply a negative stock price \( p_t \). Neither it can be that \( q_t > \phi \); the right hand side of (7) would be negative, while the left hand side can only be positive. The values \( q_t = 0 \) and \( q_t = \phi \) are feasible value but only when \( \chi > 0 \), that is, only for some specific ranges of \( \sigma \) and \( \alpha \). While this a possibility, in order to have a unique definition of equilibrium across the different parametric cases, without loss of generality, we are going to restrict to the case of \( q_t \in (0, \phi) \).

We now provide a definition of an equilibrium with respect to the divided-price ratio \( q_t \).

**Definition 1** An equilibrium is a sequence \( \{q_t\}_{t=0}^\infty \) with \( q_t \in (0, \phi) \) for any \( t > 0 \), satisfying the dynamic equation (7) and the transversality condition \( \lim_{t \to \infty} \beta^t \left[ U_c(d_t, p_t) - U_s(d_t, p_t) \right] p_t = 0 \), where \( d_t = \gamma^t d \) and \( p_t = \frac{d_t}{q_t} \).

²Since in equilibrium dividends are equal to output, this is equivalent to assuming a trend in aggregate activity. This deterministic process is not meant to capture the dividends dynamics observed in reality. For that, we would need to add a stochastic component. We are going to explore this in a later section.
Figure 1: **Equilibrium Map without the SOC.** The upward-sloping bold line corresponds to the dynamic equation (7) for the case of $\alpha = 0$, analyzed in Proposition 1.

Since we are going to establish under what conditions the equilibrium converges to zero, the unique steady state $q^*$ (see Proposition 2), to $n$–periods deterministic cycles, or instead keeps fluctuating in an erratic manner (chaotic dynamics), it is useful to provide a definition of what we mean by "cycles" and "chaos".

**Definition 2 Period-n Cycle** A value "$q$" is a point of a period-$n$ cycle if it is a fixed point of the $n$-th iterate of the mapping $f(.)$ i.e. $q = f^n(q)$, but not a fixed point of an iterate of any lower order. If "$q$" is such we call the sequence $\{q, f(q), f^2(q), ..., f^{n-1}(q)\}$ a period-$n$ cycle.

**Definition 3 Topological Chaos** The mapping $f(.)$ is topologically chaotic if there exists a set "$S$" of uncountable many initial points, belonging to its domain, such that no orbit that starts in $S$ will converge to one another or to any existing periodic orbit.

We start by establishing the benchmark result: if the spirit of capitalism is absent – $\alpha = 0$, i.e. agents do not care about their status and hence do not have direct preferences for wealth - then, provided it exists, the equilibrium is unique and features a constant dividend-price ratio. This is reminiscent of the standard Lucas’ tree result: stock price fluctuations fully reflect the underlying fundamentals. In this specific case, the only fundamentals are the exogenous dividends, and, along the equilibrium path, stock prices grow at the same rate of real dividends.\(^3\)

\(^3\)For the case of zero growth, $\gamma = 1$, the economy always features the steady state as the unique equilibrium.
Proposition 1 Suppose $\alpha = 0$, i.e. the spirit of capitalism is absent, and define $\overline{q} \equiv \frac{\gamma}{\beta} (\sigma - 1) - 1$. Then, the dynamic equation (7) reduces to $q_{t+1} = \frac{q_t}{\frac{\gamma}{\beta} (\sigma - 1) - q_t}$. Moreover, the following results hold:

a) for any $\gamma \geq 1$ when $\sigma \geq 1$, and for any $\gamma \in \left[1, \frac{1}{\beta} \right)$ when $\sigma \in (0, 1)$, the steady state, whereby $q_t = \overline{q} > 0$ for any $t \geq 0$, is the unique equilibrium;

b) for any $\gamma \geq \frac{1}{\beta} \frac{1}{\sigma - 1}$ when $\sigma \in (0, 1)$, there is no equilibrium.

Proof. By setting $\alpha = 0$ in (7) and arranging, we obtain that to be an equilibrium, a sequence $\{q_t\}_{t=0}^{\infty}$, with $q_t \in (0, \phi)$ for any $t > 0$, needs to satisfy the dynamic equation:

$$q_{t+1} = \frac{q_t}{\frac{\gamma}{\beta} (\sigma - 1) - q_t} \quad (10)$$

together with the transversality condition (5). Let $F(q_t)$ denote the right hand side of (10). To facilitate the analysis, we are going to consider two separate cases: $\sigma \geq 1$ (case 1), and $\sigma \in (0, 1)$ (case 2).

Case 1. Suppose that $\sigma \geq 1$. Notice that: 1) because of $\gamma \geq 1$ and $\beta \in (0, 1)$, (10) has a unique steady state solution, $\overline{q} \equiv \frac{\gamma}{\beta} (\sigma - 1) - 1 > 0$; 2) $F'(\overline{q}) = \frac{\gamma (\sigma - 1)}{\beta}$, that is, the steady state is dynamically unstable; 3) $q_{t+1} > 0$ if and only if $q_t < \frac{\gamma (\sigma - 1)}{\beta}$; and 4) $\lim_{q_t \to \overline{q}} F(q_t) = +\infty$. From these properties, it follows that if the initial condition is $q_0 \in \left(\overline{q}, \frac{\gamma (\sigma - 1)}{\beta}\right)$, the sequence starts moving away from the steady state, and, in particular, there exists a $t'$ such that although $q_{t'-1} < \frac{\gamma (\sigma - 1)}{\beta}$ we have that $q_{t'} > \frac{\gamma (\sigma - 1)}{\beta}$, and hence $q_{t'+2} < 0$. Clearly, such a sequence would violate the strict positivity of $q_t$: hence, there can not be an equilibrium with $q_0 > \overline{q}$.

On the other hand, for any $q_0 < \overline{q}$, the sequence $\{q_t\}_{t=0}^{\infty}$ would converge to zero: $\lim_{t \to \infty} q_t = 0$. To be an equilibrium, this path needs to satisfy the transversality condition (5). Setting $\alpha = 0$ (and hence $U_s(c_t, s_t) = 0$), and making use of the equilibrium conditions $a_t = 1$ and $c_t = d_t$, the transversality condition can be written as follows:

$$\lim_{t \to \infty} \beta^t [U_c(c_t, s_t) - U_s(c_t, s_t)] p_t a_t = \lim_{t \to \infty} \beta^t d_t^\sigma p_t$$

$$= d \lim_{t \to \infty} \left(\frac{\beta}{\gamma (\sigma - 1)}\right)^t \overline{p}_t$$

$$= \overline{p}_t$$

$$\quad (11)$$

where the second equality follows from Assumption 3 on the dividends process and from letting $\overline{p}_t \equiv \frac{D_t}{r_t}$ (the growth-adjusted stock price). Making use of this last definition, it is straightforward to write the dynamic equation (10) as follows:

$$\overline{p}_{t+1} = \frac{\gamma (\sigma - 1)}{\beta} \overline{p}_t - d$$

$$\quad (12)$$

By simple backward iteration on (12), we obtain that:

$$\overline{p}_t = \left(\frac{\gamma (\sigma - 1)}{\beta}\right)^t \overline{p}_0 - d \sum_{k=0}^{t-1} \left(\frac{\gamma (\sigma - 1)}{\beta}\right)^k$$

$$\quad (13)$$
Our transversality condition (11) can then be written as follows:

\[
\begin{align*}
\lim_{t \to \infty} \left( \frac{\beta}{\gamma \sigma - 1} \right)^t \tilde{p}_t &= \lim_{t \to \infty} \left( \frac{\beta}{\gamma \sigma - 1} \right)^t \left[ \left( \frac{\sigma - 1}{\beta} \right)^t \tilde{p}_0 - d \sum_{k=0}^{t-1} \left( \frac{\sigma - 1}{\beta} \right)^k \right] \\
&= d \left[ \tilde{p}_0 - \frac{d\beta}{\gamma \sigma - 1} \lim_{t \to \infty} t \sum_{k=0}^{t-1} \left( \frac{\beta}{\gamma \sigma - 1} \right)^k \right] \\
&= d \left[ \tilde{p}_0 - \frac{d}{\frac{\sigma - 1}{\beta} - 1} \right] \\
&= \frac{d^2 (\bar{q} - q_0)}{\bar{q} q_0} > 0
\end{align*}
\]

where the second and third line follows from simple algebra, and the fourth line uses the definition of the steady state \( \bar{q} \) as well as the fact that \( q_0 < \bar{q} \), the transversality condition does not hold. Hence, the sequence \( \{q_t\}_{t=0}^\infty \) converging to zero can not be an equilibrium. The unique equilibrium is the steady state \( \bar{q} \) which satisfies both the dynamic equation (10) and the transversality condition: \( \lim_{t \to \infty} \beta^t U_c(c_t, s_t) p_t = \lim_{t \to \infty} \left( \frac{\beta}{\gamma \sigma - 1} \right)^t \tilde{p}_t = \frac{d^2}{\bar{q}} \lim_{t \to \infty} \left( \frac{\beta}{\gamma \sigma - 1} \right)^t = 0 \).

Case 2. If \( \sigma \in (0, 1) \), the dynamic equation (12) has a positive steady state \( \bar{q} \equiv \frac{\gamma}{\beta} (\sigma - 1) - 1 \) if and only if \( \gamma \in \left[ 1, \beta \frac{1}{\gamma} \right) \). If the latter holds, the steady state is unstable and the dynamic equation (10) has the same properties seen for \( \sigma \geq 1 \). Therefore, the same result obtains: the unique equilibrium is the steady state itself.

On the other hand, for \( \gamma \geq \beta \frac{1}{\gamma} \), the dynamic equation (10) behaves differently. It is straightforward to see when \( \gamma > \beta \frac{1}{\gamma} \) that the mapping \( F(q_t) \) on the right hand side of (10) has the following properties: 1) the steady state \( \bar{q} \) is negative; 2) \( F'(q_t) > 1 \) for any \( q_t \in \left[ 0, \frac{\gamma}{\beta} (\sigma - 1) \right) \); 3) \( \lim_{q_t \to \frac{\gamma}{\beta} (\sigma - 1)} F(q_t) = +\infty \). This implies that for any initial condition \( q_0 > 0 \), there exists a \( t' \) such that although \( q_{t'-1} < \frac{\gamma}{\beta} (\sigma - 1) \) we have that \( q_{t'} > \frac{\gamma}{\beta} (\sigma - 1) \), and hence \( q_{t'+1} < 0 \). As this would violate the strict positivity of \( q_t \), for any \( \gamma > \beta \frac{1}{\gamma} \) when \( \sigma \in (0, 1) \) there can not be an equilibrium. A similar non-existence result applies for \( \gamma = \beta \frac{1}{\gamma} \). 

The properties of the dynamic map \( q_{t+1} = q_t \left( \frac{\gamma}{\beta} - q_t \right)^{-1} \) are illustrated in Figure 1. If the SOC is absent, the unstable steady state \( \bar{q} \) is the unique equilibrium. All paths starting above \( \bar{q} \) will eventually violate the restriction \( q_t > 0 \), while all paths starting below it do not satisfy the transversality condition.

### 3 Steady State and Dynamic Equilibria

As a first result, we show that, even under the SOC, the economy still displays a unique steady state equilibrium. Moreover, the SOC has a positive impact on the steady state stock price.
Proposition 2  The dynamic equation (7) has a unique non stochastic steady state \( q^* = \frac{\phi^{\frac{(\sigma-1)}{\phi+2(\sigma-1)}}-1}{\phi+2(\sigma-1)} > 0 \). Moreover, \( q^* \) is strictly decreasing in \( \alpha \).

Proof. Simply set \( q_{t+1} = q_t = q \) in (7) and solve for \( q \). We obtain \( q^* = \frac{\phi^{\frac{(\sigma-1)}{\phi+2(\sigma-1)}}-1}{\phi+2(\sigma-1)} > 0 \). Using the definition of \( \phi \equiv \frac{1-\alpha}{\alpha} \), by simple differentiation, \( \frac{\partial q^*}{\partial \alpha} < 0 \).

A direct consequence of this result is that there exists a unique balanced-growth path along which the stock price index \( p_t \) is proportional to the exogenous dividend. That is, by Assumption 3, along this equilibrium path, the stock price grows at the exogenous rate \( \gamma : p_t = \frac{d}{\phi^t} \). From the result that \( \frac{\partial q^*}{\partial \alpha} < 0 \), we can also notice that, in this equilibrium, an increase in the spirit of capitalism raises stock prices. As in our economy the stock is in fixed supply, other things equal, a higher spirit of capitalism increases the individual agents’ demand for this stock - wealth gives status and people want more of it - and, as a result, its price increases.

We now proceed to the characterization of the dynamic equilibria described by equation (7). Keeping the subjective discount factor \( \beta \) fixed, we derive analytical conditions for the existence of both cyclical and chaotic equilibria with respect to the CRRA coefficient \( \sigma \), the SOC parameter \( \alpha \) and the exogenous growth rate \( \gamma \). First of all, we state some simple properties of the composite parameter \( \chi \) that will be used in proving our results.

\[
\chi < 0 : \quad \text{for any } \alpha \in (0, 1) \text{ when } \sigma \in (0, 2) \\
\quad \quad \text{for any } \alpha < \frac{1}{\sigma-1} \text{ when } \sigma > 2
\]

\[
\chi \geq 0 : \quad \text{for any } \alpha \geq \frac{1}{\sigma-1} \text{ for } \sigma > 2
\]

To facilitate the exposition of our results, we are going to consider two separate cases: a) \( \sigma \in (0, 2] \), and b) \( \sigma > 2 \). We are going to show that, for appropriate ranges of \( \alpha \) and \( \gamma \), cyclical and chaotic equilibria are possible in case b), while they are impossible in case a).

3.1 The \( \sigma \in [0, 2] \) Case

For \( \sigma \in (0, 2] \) two types of equilibrium dynamics are possible: either the steady state equilibrium \( q^* \) (if the economy starts right on it) or a continuum of equilibria, each indexed by an initial condition \( q_0 \in (0, q^*) \) and converging to zero. Both equilibria are feasible as the dividend-price ratio remains positive and the transversality condition holds. By the definition of \( q_t \), this implies that, besides the balanced-growth path defined in Proposition 2, there exists a continuum of initial conditions, \( p_0 \), such that the stock price is explosive without violating the transversality condition. In this sense, under the spirit of capitalism, stock price bubbles are viable equilibria.

\footnote{We are going to consider the role of \( \beta \) in the sensitivity analysis.}
Proposition 3 Suppose that $\sigma \in (0, 2]$. Then, for any SOC $\alpha \in (0, 1)$ and any growth rate $\gamma \geq 1$, the economy displays a continuum of dynamic equilibria, whereby for any $q_0 \in (0, q^*)$ the equilibrium sequence $(q_t)_{t=0}^\infty$ converges to zero.

Proof. Consider (7). Define $K (q) \equiv q^\chi (q + 1)$ and $H (q) \equiv \frac{\gamma (\sigma - 1)}{\beta} q^\chi (1 - \frac{q}{q^*})$. Assume that $\sigma \in (0, 2]$ and that $\alpha \in (0, 1)$. Recalling the definition of $\chi$ in (8) and the property in (14), under this assumption, $\chi \in (-1, 0)$.

Simple algebra shows that, when $\sigma \in (0, 2]$, $K (q) \geq H (q)$ for $q \leq q^*$, where $q^*$ is the steady state defined in Proposition 2. Moreover, simple algebra shows that:

- $\lim_{q \to 0} H (q) = +\infty, \lim_{q \to -\phi} H (q) = 0$, and $H' (q) < 0$ for any $q \in (0, \phi)$.
- $\lim_{q \to 0} K (q) = +\infty, \lim_{q \to -\phi} K (q) = \phi^\chi (1 + \phi) > 0$ and $K' (q) < 0$ for any $q \in (0, \phi)$.

By its strict monotonicity, the function $K(.)$ is invertible over its entire domain. Hence, there exist a mapping $F = K^{-1} H$ such that for any $q_t \in (0, \phi)$ there exists a unique $q_{t+1} = F (q_t)$ solving (7). It is straightforward to prove that the mapping $F$ possesses the following properties: $\lim_{q_t \to 0} F (q_t) = 0$ and $\lim_{q_t \to -\phi} F (q_t) = +\infty; \lim_{q_t \to 0} F' (q_t) < 1; F (q^*) = q^*$ and $F' (q^*) > 1$. These together imply that the steady state is dynamically unstable, and in particular that for any initial condition $q_0 \in (q^*, \phi)$, there exists a $t' \geq 0$ such that $q_{t'+1} = F (q_{t'}) > \phi$; that is, the sequence $(q_t)_{t=0}^\infty$ generated by the mapping $F$ would fall outside the set $(0, \phi)$ and hence can not be an equilibrium.

On the other hand, for any initial condition $q_0 \in (0, q^*)$, the sequence $(q_t)_{t=0}^\infty$ monotonically converges to zero. It remains to verify that along this trajectory the transversality condition holds. First of all, from the dynamic equation (7), we have that:

$$
\left( \frac{q_{t+1}}{q_t} \right)^\chi = \frac{\gamma (\sigma - 1)}{\beta} \frac{1 - \frac{q_t}{q^*}}{q_t + 1} < \frac{\gamma (\sigma - 1)}{\beta}
$$

(16)

where the inequality follows from the fact that along this path $q_t < q^* < \phi$ and $q_{t+1} \geq 0$. By the definition of $q_t \equiv \frac{d_t}{p_t}$, Assumption 3 on the dividends process and letting $\nu \equiv -\chi \in (0, 1)$, the inequality (16) is equivalent to

$$
\left( \frac{P_{t+1}}{P_t} \right)^\nu < \frac{\gamma^{1+(1-\sigma)(\alpha-1)}}{\beta}
$$

(17)

that is, along this path, the stock price $p_t$ grows at a (gross) rate smaller than $\left( \frac{\gamma^{1+(1-\sigma)(\alpha-1)}}{\beta} \right)^{\frac{1}{\nu}}$. Let $\rho$ denote the gross growth rate of $p_t^\nu$. Clearly, from (17), $\rho < \frac{\gamma^{1+(1-\sigma)(\alpha-1)}}{\beta}$. Now consider the transversality condition
(5). This can be written as follows:

$$\lim_{t \to \infty} \beta^t U_c(c_t, s_t) \left[ 1 - \frac{U_c(c_t, s_t)}{U_c(c_t, s_t)} \right] p_t = \lim_{t \to \infty} \beta^t \frac{d^{(1-\alpha)(1-\sigma)-1} \alpha^{(1-\sigma)+1} p_t (1 - \frac{q_t}{\varphi})}{p_t}$$

$$< \lim_{t \to \infty} \beta^t d^{(1-\alpha)(1-\sigma)-1} \frac{\alpha(1-\sigma)+1}{p_t}$$

$$= d^{(1-\alpha)(1-\sigma)-1} \lim_{t \to \infty} \left( \frac{\beta}{\gamma(1+(1-s)(1-\alpha))} \right)^t p_t$$

$$= 0$$

where the first line follows from the utility’s functional form and the definition of $q_t$, the second line from the fact that along the path $q_t \geq 0$ and below $\varphi$, the third line from Assumption 3 on the dividends process and the fact that $\nu = \alpha (1 - \sigma) + 1$, and the fourth one from $\varphi \equiv \left( \frac{p_{t+1}}{p_t} \right)^{\nu}$. From the latter and the inequality (17), it follows that $\lim_{t \to \infty} \left( \frac{\beta}{\gamma(1+(1-s)(1-\alpha))} \right)^t \varphi = 0$ and hence the transversality condition holds. ■

This result implies that even if there exist a unique balanced growth path - whereby the dividend-price ratio remains constant - the economy features a continuum of other equilibria along which $q_t$ converges to zero, that is, the stock price grows faster than the underlying dividends. Agents’ believes about non-fundamentals factors driving stock prices are self-fulfilled in equilibrium. Even if stock prices display bubbly dynamics, along these explosive trajectories, all optimality conditions hold.

The result of Proposition 3 is illustrated in Figure 1. As a matter of fact, in this case, the properties of the equilibrium map solving (7) are essentially identical to those of $q_{t+1} = q_t \left( \frac{\gamma^{\sigma-1}}{\beta} - q_t \right)^{-1}$ related to Proposition 1. The only difference is that now the steady state is not the unique equilibrium. Any dividend-price ratio sequence starting below the steady state - and hence converging to zero - is a feasible equilibrium, since along any of those paths the transversality condition holds.

Notice that the case of utility function which is separable in consumption and status - i.e. $\sigma = 1$ - is nested in the previous Proposition, as the specification in (2)-(3) would reduce to $(1 - \alpha) \ln c_t + \alpha \ln s_t$. This corresponds to the case analyzed by Kamihigashi (2007). He shows that under the spirit of capitalism the transversality condition is not violated even if the stock price sequences diverges to infinity. As a consequence, stock price bubbles can not be ruled out based on an optimality argument, as in the standard Lucas’s tree environment. Our Proposition goes beyond Kamihigashi’s result showing that, whatever the extent of the spirit of capitalism, bubbles are possible also under a non-separable utility, as long as the coefficient of relative risk aversion remains relatively low.
3.2 The $\sigma > 2$ Case

From (14)-(15), when $\sigma > 2$ the composite parameter $\chi$ entering the dynamic equation (7) can take either sign, depending on the extent of the SOC in utility. As the next Proposition shows, if $\alpha > \frac{1}{\sigma - 1}$ and conditional on the growth rate $\gamma$ being not too high, we have well-defined forward dynamics, whereby any sequence $\{q_t\}^\infty_{t=0}$ starting at some $q_0 \in (0, 1)$ and satisfying the dynamic equation (7) is an equilibrium.

**Proposition 4** Assume that $\sigma > 2$ and $\alpha > \frac{1}{\sigma - 1}$. Moreover, define the threshold value $\gamma \equiv \left[ \frac{\beta}{(1+\phi) (1+\chi)^{1+\chi}} \right]^{\frac{1}{\sigma - 1}} > 1$. Then, as long as $\gamma < \gamma^*$, there exists a mapping $F : (0, \phi) \to (0, \phi)$ such that any path $\{q_t\}^\infty_{t=0}$ that satisfies $q_{t+1} = F(q_t)$, for any $q_0 \in (0, 1)$ is an equilibrium. Moreover, $F$ has the following properties: 1) $\lim_{q_t \to 0} F(q_t) = 0$; 2) $F$ is single-peaked at $q^* \equiv \frac{\beta}{1+\chi} \in (0, 1)$, with $F'(q_t) \geq 0$ for $q_t \leq q^*$; 3) $F(q^*) = q^*$ where $q^* = \frac{\beta}{1+\chi} > 0$ is the steady state defined in Proposition 2; 4) $F'(0) = \frac{\sigma - 1}{\sigma} > 1$.

**Proof.** Consider the dynamic equation (7), and define the functions $K(q) \equiv q^\chi (q+1)$ and $H(q) \equiv \frac{\sigma - 1}{\sigma} q^\chi \left( 1 - \frac{2}{\phi} \right)$. By (14)-(15), for $\sigma > 2$ and $\alpha > \frac{1}{\sigma - 1}$, we have that $\chi > 0$. Simple algebra shows that $K$ and $H$ have the following properties:

- $K(q) > 0$ for any $q > 0$, $\lim_{q \to 0} K(q) = 0$, $K'(q) > 0$ and $\lim_{q \to 0} K(q) = \phi^\chi (1 + \phi) > 0$;
- $H(q) \geq 0$ for $q \leq 1$, $\lim_{q \to 0} H(q) = \lim_{q \to 0} H'(q) = 0$, $H'(q) \geq 0$ for $q \leq \frac{\phi^\chi}{1+\chi} \equiv q^c$.

By its strict monotonicity, the function $K$ is invertible and hence we have a well-defined mapping $F \equiv K^{-1}$. The properties 1) - 3) of the mapping $F$ are straightforward to verify. It is immediate to verify that the feasible set $(0, \phi)$ is invariant under $F$ - hence $F : (0, \phi) \to (0, \phi)$ - as long as $F(q^c) < \phi$. This occurs when $H(q^c) < \lim_{q \to 0} K(q) = \phi^\chi (1 + \phi)$, which, after simple algebra, is equivalent to $\gamma < \left[ \frac{\beta}{(1+\phi) (1+\chi)^{1+\chi}} \right]^{\frac{1}{\sigma - 1}}$.

Now, consider a sequence $\{q_t\}^\infty_{t=0}$ which is constructed by iterating on the mapping $F$ for an initial condition $q_0 \in (0, \phi)$. For this to be an equilibrium we need to verify that the transversality condition holds. As $F : (0, \phi) \to (0, \phi)$, we must have $q_t > \varepsilon$ for any $t \geq 0$, where $\varepsilon$ is an arbitrarily small positive constant.

The transversality condition (5) can be written as follows:

$$\lim_{t \to \infty} \beta^t U_c(c_t, s_t) \left[ 1 - \frac{U_s(c_t, s_t)}{U_c(c_t, s_t)} \right] p_t = \lim_{t \to \infty} \beta^t d_t^{(1-\alpha)(1-\sigma)-1} p_t^{(1-\sigma)+1} \left( 1 - \frac{q_t}{\phi} \right)$$

$$< \lim_{t \to \infty} \beta^t d_t^{(1-\alpha)(1-\sigma)-1} p_t^{(1-\sigma)+1} (1 - \varepsilon)$$

$$= \left( 1 - \frac{\varepsilon}{\phi} \right) \lim_{t \to \infty} \beta^t d_t^{1-\sigma} q_t^\chi$$

$$= \left( 1 - \frac{\varepsilon}{\phi} \right) \phi^\chi \lim_{t \to \infty} \left( \frac{\beta}{1+\chi} \right)^t q_t^\chi$$

$$< \left( 1 - \frac{\varepsilon}{\phi} \right) \phi^\chi d_t^{1-\sigma} \lim_{t \to \infty} \left( \frac{\beta}{1+\chi} \right)^t = 0$$
where the first line follows from the utility’s functional form and the definition of \( q_t \), the second line from \( q_t > \varepsilon > 0 \), the third line from the definition of \( q_t \) and of \( \chi \), the fourth line from Assumption 3 on the dividends process, and the fifth one from the fact that \( q_t \in (0, \phi) \) and \( \chi > 0 \). But since, for \( \sigma > 1 \), \( \frac{\gamma^{\sigma-1}}{\beta} > 1 \), the last limit must be zero. We can then conclude that the transversality condition always holds, and hence that any sequence \( \{q_t\}_{t=0}^{\infty} \) which satisfies the mapping \( F \), for any initial condition \( q_0 \in (0, \phi) \), is an equilibrium. ■

Building on this result, we can show that cyclical and chaotic dynamics occur as the economy grows sufficiently fast.\(^5\)

**Proposition 5** Suppose that \( \sigma > 2 \) and that \( \alpha > \frac{1}{\sigma-1} \). Then there exist threshold values \( \gamma^c \in (1, \tau) \) and \( \gamma^h \in (\gamma^c, \tau) \) such that:

1. period-2 cycles exist for \( \gamma > \gamma^c \)
2. period-3 cycles and hence chaotic dynamics exist for \( \gamma > \gamma^h \).

**Proof.** The proof of 1. involves searching for a flip bifurcation at the interior steady state \( q^* \). Define the auxiliary function \( M(q_t) = q_t - F^2(q_t) \). A period-2 cycle is given by the solutions to \( M(q_t) = 0 \). By the properties if the mapping \( F \) spelled in Proposition 4 it follows that \( \lim_{q_t \to 0} M(q_t) = 0 \) and \( \lim_{q_t \to \infty} M(q_t) > 0 \).

Moreover, by chain rule, \( \lim_{q_t \to 0} M'(q_t) = 1 - \left[ \lim_{q_t \to 0} F'(q_t) \right]^2 < 0 \). Hence, by continuity of the function \( M(q_t) \) over the range \( (0, \phi) \), a sufficient condition for the existence of a period-2 cycle is that \( M'(q^*) < 0 \) that is \( F'(q^*) < -1 \). By the definition of the mapping \( F \) in Proposition 4, we have that:

\[
F'(q^*) = \frac{\gamma^{\sigma-1}\chi \left( 1 - \frac{q^*}{\phi} \right) - q^*}{\beta \chi (1 + q^*) + q^*} = (1 + \delta) \frac{\chi (1 + \phi) - \delta}{\chi (1 + \delta + \phi + \delta \phi) + \delta \phi}
\]

where we have defined \( \delta \equiv \frac{\gamma^{\sigma-1}}{\beta} - 1 \) and written the steady state as \( q^* = \frac{\alpha \phi}{\tau + \beta + \tau} \). Since \( \gamma \geq 1 \) and \( \beta \in (0, 1) \), then \( \delta > 0 \). The condition \( F'(q^*) < -1 \) is then equivalent to \( \delta^2 + \delta \left( 1 - \phi - 2\chi (1 + \phi) \right) - 2\chi (1 + \phi) > 0 \).

Simple algebra shows that, under the restriction \( \delta > 0 \), this inequality holds if and only if:

\[
\delta > \frac{-[1 - \phi - 2\chi (1 + \phi)] + \sqrt{(1 - \phi)^2 + 4\chi (1 + \phi)^2 (1 + \chi)}}{2} \equiv \delta^c \quad (18)
\]

Recall from Proposition 4 that for the feasible set \( (0, \phi) \) to be invariant under \( F \) we required that \( \gamma < \tau \). By our definition of \( \delta \), the restriction \( \gamma < \tau \) can be equivalently written as \( \delta < (1 + \phi) \frac{(1 + \chi)^{1+\chi}}{\chi^{\frac{1+\chi}{\chi}}} - 1 \equiv \delta^c \). It follows

\(^5\)Cycles can still occur for growth rates above the threshold \( \tau \) defined in Lemma ?? However, the set of initial values \( q_0 \) for which the economy would converge to those cycles would not be of full measure. This is because for \( \gamma > \tau \) the equilibrium set \( (0, 1) \) is not invariant under the mapping \( F \) anymore, which now maps from \( (0, \phi) \) to \( \mathbb{R}^{++} \). Hence for some \( q_t \in (0, \phi) \) the map would give a \( q_{t+1} > \phi \).
that the set of positive values of \( \delta \) (and hence of \( \gamma \geq 1 \)) which satisfies the inequality (18) is non-empty if and only if \( \delta^c < \delta^u \).

Consider \( \delta^c \). Extensive algebra shows that \( \frac{(1+\chi)^{1+\chi}}{\chi} - 1 > 2\chi \) for any \( \chi > 0 \). Given that \( \phi \equiv \frac{1-a}{a} > 0 \), it immediately follows that \( (1 + \phi) \frac{(1+\chi)^{1+\chi}}{\chi} > (1 + \phi) (1 + 2\chi) \), which, by our definition of \( \delta^c \), is equivalent to \( \delta^c > (1 + \phi) (1 + 2\chi) - 1 \). Since, as simple algebra shows, \( \delta^c < (1 + \phi) (1 + 2\chi) - 1 \), we have that \( \delta^c < \delta^u \).

We can then conclude that period-2 cycles exists for \( \delta^c < \delta < \delta^u \), which by \( \delta \equiv \frac{2^{(\sigma-1)}}{\beta} - 1 \), it requires the growth rate \( \gamma \) to satisfy the following inequality: \( \gamma^c < \gamma < \gamma^u \) for \( \gamma^c \equiv \left[ \beta (1 + \delta^c) \right]^{\frac{1}{1+\sigma}} \). This concludes the proof of 1.

The proof of the existence of chaotic dynamics consists in showing that there exists a threshold growth rate \( \gamma^h \in (\gamma^c, \gamma^u) \) such that if \( \gamma > \gamma^h \) there exists period-3 cycles, which, by Sarkovskii (1964) and Li and Yorke (1975), is a sufficient condition for the existence of chaotic dynamics. To do that, define the auxiliary function \( N(q_t) = q_t - F^3(q_t) \). First of all, \( \lim_{q_t \to 0} N(q_t) = 0 \), \( \lim_{q_t \to \phi^-} N(q_t) = 1 - \lim_{q_t \to \phi^-} F^3(q_t) = 1 \) and \( N(q^*) = 0 \). Moreover, \( N'(q^*) = 1 - \left(F'(q^*)\right)^3 \). Since for \( \gamma > \gamma^c \), \( F'(q^*) < -1 \), it must be that \( N'(q^*) > 0 \). Now let’s suppose that \( \gamma = \gamma^u \). From the properties of the mapping \( F \) in Proposition 4 we have that \( N(q^c) = q^c > 0 \). By the continuity of \( F \) with respect to the growth rate \( \gamma \), this must also hold for some values of \( \gamma \) in the left neighborhood of \( \gamma^u \). More specifically, there must be a \( \gamma^h \in (\gamma^c, \gamma^u) \) such that \( N(q^h) = q^h > 0 \) for any \( \gamma^h < \gamma < \gamma^u \). Notice that this is sufficient for the existence of a point \( q^* \in (0, q^*) \) such that \( N(q^*) = 0 \), i.e. \( q^* = F^3(q^*) \) : a period-3 cycle exists. Therefore, by the Sarkovskii (1964) and Li and Yorke (1975) theorems, cycles of any period \( n \) (for \( n \) integer) as well as aperiodic chaotic fluctuations exist as well.

From Proposition 4 we know that both the cyclical and the chaotic paths are equilibria since along those paths \( q_t \in (0, \phi) \) and hence the transversality condition is always satisfied. ■

According to Proposition 5, it is the combination of real growth and the SOC that can induce cyclical and chaotic endogenous fluctuations. More specifically, an extreme degree of SOC in the economy might not be sufficient to induce complex stock price dynamics if dividends’ growth was too small. From the related proof, one can also see that the period-2 cycles threshold, \( \gamma^c \), is strictly increasing in the subjective discount rate \( \beta \). As agents get more impatient, period-2 cycles (and then chaos) can occur at lower growth rates. Interestingly, they would even occur in a zero-growth economy, i.e. \( \gamma = 1 \), provided that \( \beta < \frac{1}{1+\sigma} \), where \( \delta^c \) defined by (18).

In addition to this, notice that the threshold \( \alpha^m \) is strictly decreasing in the CRRA coefficient \( \sigma \), meaning that the case of Proposition 5 occurs for lower degrees of the SOC as we consider economies with a higher risk aversion. For instance, it requires \( \alpha > 0.66 \) if we set \( \sigma = 2.5 \), but only \( \alpha > 0.14 \) for \( \sigma = 8 \).

As we are going to see next, the possibility of endogenous dynamics for the case of \( \alpha < \frac{1}{\delta^c} \) is more restrictive. First of all, the equilibrium dynamics are not well-defined in a forward sense. That is, for a given
Figure 2: Equilibrium maps under the SOC for $\sigma > 2$ and $\alpha > (\sigma - 1)^{-1}$. The figure displays the equilibrium mapping $q_{t+1} = F(q_t)$ solving the dynamic equation (7) for the case of $\chi > 0$. Panels a) and b) correspond to the case of $\gamma < \gamma^c$ (the period-2 cycle bifurcation threshold), whereby the steady state is dynamically stable and convergence towards it is either monotonic or cyclical. Panel c) corresponds to the case of $\gamma > \gamma^c$ for which periodic equilibria and chaos can occur. Panel c) shows a period-2 $(q_L, q_H)$ as well as a period-3 $(q_1, q_2, q_3)$ cycle.

$q_t \in (0, \phi)$, we have either two (i.e. a correspondence rather than a map) or no $q_{t+1} \in (0, \phi)$ solving the dynamic equation (7). On the contrary, for any given $q_{t+1} \in (0, \phi)$ there exists a unique $q_t \in (0, \phi)$ solving (7), hence a well-defined backward map from future (expected) to current dividend-price ratio. Second, period-2 cycles occur for intermediate values of the growth rate $\gamma$, but only if the CRRA coefficient $\sigma$ is larger than 3. Third, it is not possible to establish analytical conditions for the existence of period-3 cycles, and hence chaotic dynamics. For this, we have to resort to numerical simulations.

Proposition 6 Suppose that $\sigma > 2$ and $\alpha < \frac{1}{\sigma-1}$. Then, there exist well defined backward dynamics, that is a mapping $G: (0, \phi) \to (0, \phi)$ such that any path $\{q_t\}_{t=0}^{\infty}$ that satisfies $q_t = G(q_{t+1})$ is an equilibrium. Moreover:

1. for any $\alpha < \frac{1}{\sigma-1}$ when $\sigma \in (2, 3]$ and for any $\alpha < \frac{\sigma + \sqrt{2(\sigma - 1)}}{2(1+\sigma)}$ when $\sigma > 3$: the backward-dynamics converge to the unique steady state $q^*$ for any growth rate $\gamma \geq 1$;

2. for any $\alpha \in \left(\frac{\sigma + \sqrt{2(\sigma - 1)}}{2(1+\sigma)}, \frac{1}{\sigma-1}\right)$ when $\sigma > 3$: there exist threshold values $\gamma^L$ and $\gamma^H$, such that period-2 cycles occur for $\gamma \in (\gamma^L, \gamma^H)$.

Proof. Consider the dynamic equation (7) and define the functions $K(q) \equiv q^\chi (q + 1)$ and $H(q) \equiv \frac{\gamma^{(\sigma-1)} - q^\chi}{\beta - q^\chi} \left(1 - \frac{\sigma}{\phi}\right)$. By (14)-(15), for $\sigma > 2$ and $\alpha < \frac{1}{\sigma-1}$, we have that $\chi < 0$. Let $v \equiv -\chi \in (0, 1)$. Simple
algebra shows that, for the case considered here, \( K(q) \geq H(q) \) for \( q \leq q^* \), where \( q^* \) is the steady state defined in Proposition 2, together with:

- \( \lim_{q \to 0} H(q) = +\infty \), \( \lim_{q \to 1} H(q) = 0 \), \( H(q) > 0 \) and \( H'(q) < 0 \) for any \( q \in (0, \phi) \);
- \( \lim_{q \to 0} K(q) = +\infty \), \( \lim_{q \to 1} K(q) = \phi^N(\phi + 1) > 0 \), \( K'(q) \geq 0 \) for \( q \leq \frac{v}{\phi(\phi + 1)} \equiv q^k \), and \( q^k \in (0, 1) \).

Being non-monotonic, the function \( K \) is not invertible and hence there is not a well-defined mapping from \( q_t \) to \( q_{t+1} \) solving the dynamic equation (7). Instead, for a given \( q_t \in (0, \phi) \) there exists either none or two values for \( q_{t+1} \in (0, \phi) \) solving (7). On the contrary, \( H \) is always invertible on the domain \((0, \phi)\), which guarantees the existence of a well-defined backward-mapping \( G : (0, \phi) \to (0, \phi) \) for which \( q_t = G(q_{t+1}) \) solves (7). Moreover, \( G \) has the following properties: \( \lim_{q_{t+1} \to 0} G(q_{t+1}) = 0 \), \( \lim_{q_{t+1} \to \phi} G(q_{t+1}) = q^* \), \( \lim_{q_{t+1} \to 1} G'(q_{t+1}) > 1 \), \( G'(q_{t+1}) \geq 0 \) for \( q_{t+1} \leq q^k \) with \( G'(q^*) < 1 \).

Now, let \( \delta \equiv \frac{2(\sigma - 1)}{\beta} - 1 \), and notice that \( \delta > 0 \) since \( \gamma \geq 1 \), \( \sigma > 2 \) and \( \beta \in (0, 1) \). By the Implicit Function Theorem:

\[
G'(q^*) = \frac{K'(q^*)}{H'(q^*)} = \frac{q^* (1 - v) - v}{(1 + \delta)[v - q^* (1 - v)]}
\]

where, the steady state \( q^* \) defined in Proposition 2 can be written as \( q^* \equiv \frac{\phi^\delta}{\phi + \phi^\delta} \).

Define the auxiliary function \( M(q) = q - G^2(q) \). A period-2 cycle for the backward map \( G \) is the solutions to \( M(q) = 0 \). By the properties of \( G \), it easily follows that \( \lim_{q \to 0} M(q) = 0 \) and \( \lim_{q \to \phi} M(q) > 0 \). Moreover, by chain rule, \( \lim_{q \to 0} M'(q) = 1 - \left[ \lim_{q \to 0} G'(q) \right]^2 < 0 \). Hence, by continuity of the function \( M(q) \) over the range \((0, \phi)\), a sufficient condition for the existence of a period-2 cycle is that \( M'(q^*) < 0 \) that is \( G'(q^*) < 1 \). On the other hand, if \( G'(q^*) > -1 \) (and since \( G'(q^*) < 1 \)) the steady state is dynamically stable, in backward sense.

Simple algebra shows that \( G'(q^*) < -1 \) if and only if:

\[
\delta^2 + \delta [1 - \phi + 2v(1 + \phi)] + 2v(1 + \phi) < 0
\]

We are going to consider two separate cases: 1) \( \sigma \in (2, 3) \), and 2) \( \sigma > 3 \).

Suppose that \( \sigma \in (2, 3) \). Notice that the first and the third term on the left hand side of (20) are always positive. Using the definitions \( \phi \equiv \frac{1 - \alpha}{\alpha} \) and \( v = -\chi = 1 + (1 - \sigma) \alpha \), after simple algebra, one can verify that the second term \( [1 - \phi + 2v(1 + \phi)] \geq 0 \) for \( \alpha \leq \frac{1}{2(\sigma - 2)} \). Since, in this proposition, we are restricting to \( \alpha < \frac{1}{\sigma - 2} \) and since \( \frac{1}{\sigma - 2} \leq \frac{1}{2(\sigma - 2)} \) for \( \sigma \in (2, 3) \), then \( \alpha < \frac{1}{2(\sigma - 2)} \) and hence \( [1 - \phi + 2v(1 + \phi)] > 0 \). But then the inequality (20) never holds, and as a consequence it must be that \( G'(q^*) \in (-1, 1) \) : the steady state is dynamically stable in backward-dynamics. For any \( q_0 \in (0, \phi) \), the sequence \( q_t \) converges to \( q^* \) for \( t \to -\infty \). 

16
Now suppose $\sigma > 3$. Still we have that the first and the third term on the left hand side of (20) are positive, and that $[1 - \phi + 2\nu (1 + \phi)] \leq 0$ for $\alpha \leq \frac{1}{2(\sigma - 2)}$. However, now $\frac{1}{2(\sigma - 2)} < \frac{1}{\sigma - 1}$. Clearly, for $\alpha \leq \frac{1}{2(\sigma - 2)}$, we have that $[1 - \phi + 2\nu (1 + \phi)] \geq 0$, implying that the necessary and sufficient condition for $G'(q^*) < -1$ (20) never holds. If that is the case, the same result of Point 1 obtains.

If instead we consider $\alpha \in \left(\frac{1}{2(\sigma - 2)}, \frac{1}{\sigma - 1}\right)$, the term $[1 - \phi + 2\nu (1 + \phi)] < 0$ and the could be some values of $\delta$ for which (20) may hold. First, notice that the related quadratic equation - that is, (20) holding with equality - admits real solutions as long as its discriminant is non-negative. Simple algebra shows that such discriminant is equal to $\Delta = (1 - \phi)^2 + 4\nu (\nu - 1) (1 + \phi)^2$. By the definitions of $\nu$ and $\phi$ again, one can verify that $\Delta \geq 0$ for $\alpha \geq \frac{\sigma + \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)}$ and for $\alpha \leq \frac{\sigma - \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)}$. However, $\frac{\sigma - \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)} < \frac{1}{2(\sigma - 2)}$ (for $\sigma > 2$) and hence we do not need to consider the case of $\alpha \leq \frac{\sigma - \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)}$. On the other hand, $\frac{\sigma + \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)} \in \left(\frac{1}{2(\sigma - 2)}, \frac{1}{\sigma - 1}\right)$, falling inside the range of values for $\alpha$ we are considering. For $\alpha = \frac{\sigma + \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)}$, the quadratic equation does not have real solution, implying that the quadratic inequality (20) never holds: hence, no period-2 cycles in this case. For $\alpha = \frac{\sigma + \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)}$, we have that $\Delta = 0$ : the quadratic equation has two coincident solutions, but no $\delta$ exists that satisfies the inequality in (20). For $\alpha \in \left(\frac{\sigma + \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)}, \frac{1}{\sigma - 1}\right)$, the quadratic equation has two distinct (and positive) real solutions, $\delta^L$ and $\delta^H$, such that the inequality holds for any $\delta \in (\delta^L, \delta^H)$.

By the definition of $\delta$, this implies that for the case of $\delta > 3$ and $\alpha \in \left(\frac{\sigma + \sqrt{2(\sigma - 1)}}{2(1 + (1 - \sigma)^2)}, \frac{1}{\sigma - 1}\right)$, there exist period-2 cycles for growth rates within the range $(\gamma^L, \gamma^H)$, where $\gamma^i \equiv [\beta (1 + \delta^i)]^{1 \over \sigma - 1}$ for $i = L, H$. ■

We perform a simple quantitative evaluation of the theoretical results presented in Propositions 5 and 6. We fix $\beta = 0.92$, which corresponds to an annual net interest rate of about 8.5%. This is indeed higher than what observed in most western economies. But once again, we think of this model as being more suitable for emerging economies moving towards a capitalist market structure and undergoing major social transformations, like China, India and other Asian economies, as well as most Latin American countries. Towards the end of this section, we are going to explore the implications of a higher discount rate.

Figure 3 plots the orbit bifurcation diagram for the case of $\sigma = 2.5$ - for which the threshold SOC is $\frac{1}{\sigma - 1} = 0.66$ - setting $\alpha = 0.69$. This corresponds to the case of Proposition 5. As the figure shows, if the growth rate remains sufficiently low (below 1.047 in this numerical example) the dividend-price ratio $q_t$ converges to the unique steady state for any initial condition $q_0 \in (0, \phi)$. The multiplicity of viable initial conditions implies that the equilibrium is globally indeterminate. However, no cyclical or chaotic dynamics appear, that is, although along the transition the dividend-price ratio reflects non-fundamental factors, in the limit it converge to the unique steady state, whose value is determined by fundamentals only.

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Figure 3: **Orbit bifurcation diagram related to Proposition 5.** The figures displays the orbit bifurcation diagram with respect to the dividends growth rate $\gamma$ for the case of $\sigma > 2$ and $\alpha > (\sigma - 1)^{-1}$. Calibration: $\sigma = 2.5$, $\alpha = 0.69$ and $\beta = 0.92$.

As we consider values of $\gamma$ above the threshold $\gamma^c = 1.047$, period-2 cycles appear: that is, for any initial condition, the dividend-price ratio will eventually converge to an equilibrium characterized by a deterministic switch between a low and a high value. Notice that neither of this two values is the steady state, which means that, even in the limit, the dividend-price ratio is affected by non-fundamentals. If $\gamma$ is raised further, higher order cycles appear (period-4, period-6, etc...cycles) till, the equilibrium dynamics do not seem to converge to any periodic equilibrium but instead display chaotic behavior.

Figure 4 considers the case of a lower spirit of capitalism, as in Proposition 6. Based on those results, we fix $\sigma = 4$ (we need $\sigma > 3$) and $\alpha = 0.33$ which satisfies the restriction $\alpha \in (\alpha^l, \alpha^m)$ stated in the proposition. We obtain that the economy converges to the unique steady state (in a backward sense) for gross growth rates which are either very low (almost equal to 1) or above 1.18. For all other values, period-2 cycles appear, consistently with the theoretical result stated in the proposition. However, we do not find any evidence of chaotic dynamics.\footnote{As a matter of fact, we do not find chaotic dynamics for any parametrization that satisfied the conditions stated in Proposition 6.}

One might wonder what is the role of the discount factor $\beta$ in all this. This is shown in Figure for the case of $\beta = 2.5$ and $\alpha = 0.69$, which is the parametrization used in Figure 1. As discussed, for a discount rate $\beta = 0.92$ cyclical equilibria start occurring for $\gamma > 1.047$. For this reason, we fix $\gamma = 1.03$, such that we have convergence to the unique steady state for the benchmark $\beta = 0.92$. Clearly, impatience is another source of volatile dynamics. For a discount rate approaching 1 (agents are very patient), the dividend-price...
Figure 4: Orbit bifurcation diagram related to Proposition 6. The figures displays the orbit bifurcation diagram with respect to the dividends growth rate $\gamma$ for the case of $\sigma > 2$ and $\alpha < (\sigma - 1)^{-1}$. Calibration: $\sigma = 4$, $\alpha = 0.33$ and $\beta = 0.92$.

ratio converges to the unique steady state. As $\beta$ is lowered (in this case below 0.9), period-2 cycles first and higher order cycles/chaotic dynamics then emerge.

4 Uncertainty and Recurrent Bubbles

We extend our analysis to the case of aggregate uncertainty by assuming that the dividends growth rate is stochastic. Similarly to Lansing (2010) we let $x_t \equiv \ln \left( \frac{d_t}{s_{t-1}} \right)$ and assume that $x_t$ is generated by the following stationary stochastic process:

$$x_t - \bar{x} = \rho (x_{t-1} - \bar{x}) + \varepsilon_t, \quad \text{for } \varepsilon_t \sim iidN \left( 0, \sigma^2 \right)$$  \hspace{1cm} (21)

where $|\rho| < 1$ and $\bar{x} > 0$. That is, $x_t$ is normally distributed with unconditional mean and variance equal to, respectively, $\bar{x}$ and $\frac{\sigma^2}{1-\rho^2}$. Under this assumption, the dividends gross growth rate $\gamma_t \equiv \frac{d_t}{s_{t-1}}$ is then log-normal with $E(\gamma_t) = \exp \left( \bar{x} + \frac{1}{2} \frac{\sigma^2}{1-\rho^2} \right)$ and $Var(\gamma_t) = \exp \left( \frac{\sigma^2}{1-\rho^2} - 1 \right) \exp \left( 2\bar{x} + \frac{\sigma^2}{1-\rho^2} \right)$. We are going to refer to $\varepsilon_t$ as the fundamental shock.

Solving the representative agent’s optimization problem under uncertainty, we obtain that the equilibrium dynamics of the dividend-price ratio $q_t$ are described by the following non-linear stochastic difference equation:

$$E_t \left[ \gamma_{t+1}^{1-\sigma} q_{t+1} \left( 1 + q_{t+1} \right) \right] = \frac{q_t^{\chi}}{\beta} \left( 1 - \frac{q_t}{\phi} \right)$$  \hspace{1cm} (22)
Figure 5: **Orbit bifurcation diagram and impatience.** The figures displays the orbit bifurcation diagram with respect to the subjective discount rate $\beta$ for the case of $\sigma > 2$ and $\alpha > (\sigma - 1)^{-1}$. Calibration: $\sigma = 2.5$, $\alpha = 0.69$ and $\gamma = 1.03$.

The latter can be written as:

$$\gamma_{t+1}^{1-\sigma} q_{t+1}^\chi (1 + q_{t+1}) = \frac{q_t^\chi}{\beta} \left(1 - \frac{q_t}{\phi}\right) (1 + \nu_{t+1})$$

or equivalently,

$$\frac{q_{t+1}^\chi (1 + q_{t+1})}{\gamma_{t+1}^{\sigma - 1} (1 + \nu_{t+1})} = \frac{q_t^\chi}{\beta} \left(1 - \frac{q_t}{\phi}\right)$$

where $\nu_{t+1}$ is a arbitrary *iid* expectational error with $E_t(\nu_{t+1}) = 0$ and $\gamma_{t+1} = \exp(x_{t+1})$ with $x_{t+1}$ generated by (21). The expectational error $\nu_{t+1}$ could be correlated or uncorrelated with the *iid* fundamental shock $\varepsilon_t$. If uncorrelated, $\nu_{t+1}$ is a pure sunspot shock. We are going to focus on the case of $\chi > 0$, for which we have shown that, in the deterministic environment, the equilibrium path converges either to the steady state or to period-$n$ cycles, or becomes chaotic.

Kamihigashi (2007) shows that it is possible to obtain a stock-price path which resembles a bubble, where the boom and bust phases are completely endogenous, if the realizations of the sunspot shock are sufficiently large. The same result would hold in our model for the case of $\sigma \in (0, 2]$, since, as shown in Section 3.1, our deterministic stock price path is monotonically increasing, as obtained by Kamihigashi (2007).

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8Alternatively, one could introduce the expectational error in the standard additive form: $\gamma_{t+1}^{1-\sigma} q_{t+1}^\chi (1 + q_{t+1}) = \frac{q_t^\chi}{\beta} \left(1 - \frac{q_t}{\phi}\right) + \nu_{t+1}$, where $E_t\nu_{t+1} = 0$. Our multiplicative formulation is similar to the one used by Evans (1991) and Kamihigashi (2007).
More generally, the assumption of very volatile and even explosive non-fundamental shocks has been often used in consumption-based asset pricing models in order to generate stock price dynamics that resemble a bubble. In linear (or linearized) models, this has been accomplished by splitting the equilibrium stock price $p_t$ between a fundamental dividends-driven component ($p_f^t$) and a non-fundamental factor ($p_b^t$), with the latter being any process satisfying the minimal restriction $E_t p_{t+1}^b = \beta^{-1} p_t^b$. The explosiveness of the $p_b^t$ process is a natural way to make the equilibrium stock price $p_t^t$ diverge from the underlying fundamentals. A clear disadvantage of this approach is that the bubble never crashes unless we assume it does with some exogenous probability, similarly to Blanchard (1979). However, as Diba and Grossman (1988) show, a bubble that crashes can never restart, a result that virtually makes these models incapable of generating the recurrent boom-bust cycles that we observe in the real world.

Based on the results by Evans (1991), Kamihigashi (2010) shows that it is still possible to induce recurrent bubbles in a Lucas’s tree model by adding more structure to the process of $p_b^t$, in particular, by assuming that the stochastic properties of the bubble component are related to an exogenous confidence factor.

In what follows, it is shown that our model is capable of generating recurrent bubbles in a completely endogenous fashion even though the expectational errors are kept arbitrarily small. This result is due to the interaction between a) the non-linearity of the map $F$ described in Proposition 4 for the economy without uncertainty and b) the time variation in the dividends growth rate $\gamma_t$. To capture this mechanism, let $\bar{\gamma}$ be the average growth rate for dividends, and suppose that, for a given calibration of the structural parameters, $\bar{\gamma}$ takes a value below $\gamma^c$, the period-2 cycle bifurcation threshold of the deterministic environment defined in Proposition 5). If there was no uncertainty - i.e. $\varepsilon_t$ for every $t \geq 0$ and $x_{-1} = \bar{x}$ - then $\gamma_t = \bar{\gamma}$ and, for any initial condition, the dividend-price ratio $q_t$ would converge to the unique steady state with probability one, either cyclically or monotonically. With the noisy shock $\varepsilon_t$ added, $\gamma_t$ fluctuates around $\bar{\gamma}$, more or less persistently depending on the value assigned to $\rho$. If the realizations of $\varepsilon_t$ are sufficiently small, the equilibrium will permanently fluctuate around the steady state $q^*$, but the latter will never be reached since $\gamma_t$ will never settle to $\bar{\gamma}$. On the other hand, if we allow for a larger support for $\varepsilon_t$ and/or sufficient persistence (higher $\rho$), $\gamma_t$ might take values significantly above or below $\bar{\gamma}$, which amplify the volatility of $q_t$ within the feasible set $(0,\phi)$. We are then likely to observe a sequence $\{q_t\}_{t=0}^\infty$ that fluctuates around its long run mean for many period, suddenly falls close to zero, and then reverts back to the mean, with this switch happening repeatedly at irregular intervals. The result is a price-dividend ratio $(q_t^{-1})$ displaying endogenous boom-bust phases which resemble recurrent bubbles.

Figure 6 displays a simulated equilibrium path for the price-dividend ratio $q_t^{-1}$, using the dynamic equation (23), assuming that the dividends growth rate is constant while the agents make iid expectational

\[\footnote{For $\bar{\gamma} < \gamma^c$ the slope of the mapping $F$ is smaller than on in absolute value, i.e. the steady state is dynamically stable. The converge would be monotonic or cyclical depending on whether $F'(q^*)$ is positive or negative.} \]
Figure 6: Simulated price-dividend ratio under iid sunspot shock. The four panels differ with respect to the support of the sunspot shock $\nu_{t+1}$. Panel a) corresponds to the deterministic case, for which a period-2 cycle occurs. Calibration: $\sigma = 2.5$, $\alpha = 0.69$, $\beta = 0.92$ and $\gamma = 1.05$ (above period-2 bifurcation threshold).

errors. The calibration is identical to what used before to generate the orbit bifurcation diagram of Figure 3 for the deterministic case: namely $\sigma = 2.5$ and $\alpha = 0.69 > \frac{1}{\sigma - 1}$ such that $\chi > 0$ - and $\beta = 0.92$. The mean $\bar{\nu}$ has been chosen such that the mean $\bar{\gamma} = E(\gamma_t) = 1.05$, which is above the period-2 bifurcation threshold $\gamma^c$. Absent uncertainty ($\nu_{t+1} = 0$ for every $t$), the price-dividend ratio would eventually settle on a period-2 cycle (panel a)). The introduction of a sunspot error makes the fluctuations both aperiodic and persistent, with the size of this effect depending on the sunspot’s support. Panels c) and d) for instance show that with a support equal to, respectively, $(-0.05, +0.05)$ and $(-0.1, +0.1)$ the price-dividend ratio shows several boom-bust phases, with sudden and short-lived run-ups. But for most of the time frame the simulated series develops through frequent less pronounced swings, which are due to the fact that the non-linear equilibrium map $F$ generates period-2 cycles.

The simulation in Figure 7 corresponds to the case of a iid dividends growth rate. To stress the direct implication of fundamental uncertainty, we have assumed that the agent’s expectational errors are zero at all times.$^{10}$ The four panels display the effect of increasing the volatility of the fundamental shock $\varepsilon_t$. Although

$^{10}$This is clearly unrealistic. Our next simulations relax this assumption.
the results are similar to what obtained for the case of a sunspot shock, what is even more shocking here is that although the fundamentals are iid distributed and not much volatile, the equilibrium price-dividend ratio is highly volatile and persistent. Interestingly, the fundamental shock generates more frequent boom-bust phases than the sunspot shock.

Figure 8 assesses the role of persistence of the dividends-growth rate, $\rho$, while assuming that $\sigma_\varepsilon = 0.01$ and that the sunspot shock lies within the support $(-0.01, 0.01)$. The mid-panel is identical to panel c) of Figure 7. By comparing it with the left-hand-side and the right-hand-side panels one can infer the followings. First, with $\rho > 0$ (persistent dividend growth), over the 100 periods considered, there is only one significantly large deviation from the long run mean - hence sizable boom-bust phases are less frequent. Second, with $\rho < 0$ (antipersistent dividend growth), these phases become actually more frequent and larger in size.

Our next experiment assesses whether similar dynamics can be obtained for a lower mean growth rate $\bar{\gamma}$. The three panels in Figure ?? correspond, respectively, to $\bar{\gamma}$ equal to 1.01, 1.03 and 1.05. Under the first two
Figure 8: **Simulated price-dividend ratio and persistence.** The three panels differ with respect to the persistence parameter $\rho$ of the $x_t$ process. The simulations include the sunspot shock $\nu_{t+1} \in (-0.01, 0.01)$. Calibration: $\sigma = 2.5$, $\alpha = 0.69$, $\beta = 0.92$ and $\bar{\gamma} = 1.05$.

cases, no equilibrium cycle arises in the deterministic case - i.e. absent uncertainty, the price dividend ratio would converge to the unique steady state - while the last case is identical to what displayed in Panel c) in the previous Figure. Clearly, lowering the mean growth rate does not rule out the recurrence of boom-busts phases. However the latter now appear less dramatic but more persistent. In particular, notice that the price-dividend ratio in the mid-panel does not appear unrealistic at all, compared to what observed in the United States post WWII.¹¹

We conclude our quantitative analysis by showing the effect of enlarging the volatility of the fundamental shock. We do that for the case of $\bar{\gamma} = 1.01$ of Panel a) in the previous Figure. Not surprising, a large fundamental uncertainty amplifies both the significance and the frequency of occurrence of the boom-bust phases.

From these simulations one might conclude the followings. First of all, it is not necessary to assume large non-fundamental sunspot shocks in order to induce high volatility into the model. Because of the intrinsic non-linearity, our model can generate endogenous and recurrent boom-bust cycles even if the economy is hit by *iid* fundamental shocks. Second, a persistent dividends-growth process is capable of reducing the frequency of these cycles while amplifying their size. Third, recurrent bubbles can also occur at mean growth rates for which, in a deterministic environment, the economy would simply converge to the steady state. Interestingly, in this case, the equilibrium price-dividend ratio dynamics appear rather realistic.

Figure ?? provides a simple graphical description of how the dynamic equation (23) can generate recurrent boom-bust phases. To simplify our argument, let’s suppose that the sunspot expectational shocks are zero at all times, i.e. $\nu_{t+1} = 0$, and that the dividends growth rate $\gamma_t$ switches (stochastically) between a high ($\gamma_H$)

¹¹As a matter of fact, the mean dividends (or earnings) growth rate post WWII is about 3%.
Figure 9: Simulated price-dividend ratio for different mean growth rates. The panels differ with respect to the mean growth rate $\bar{\gamma} = E(\gamma_t)$. In both panel a) and b), $\bar{\gamma}$ is below the period-2 bifurcation threshold of the deterministic case. Panel c) - for which $\bar{\gamma} = 1.05$ - corresponds to panel c) of Figure 8. Calibration: $\sigma = 2.5$, $\alpha = 0.69$, $\beta = 0.92$, $\rho = 0.5$ and $\sigma_\varepsilon = 0.01$.

Figure 10: Simulated price-dividend ratio and fundamental volatility. The three panels differ with respect to the standard deviation of the fundamental shock, $\sigma_\varepsilon$. Calibration: $\sigma = 2.5$, $\alpha = 0.69$, $\beta = 0.92$, $\rho = 0.5$ and $\bar{\gamma} = 1.01$. 

25
and a low ($\gamma_L$) value. This case corresponds to panel a) in the Figure. The hump-shaped curve corresponds to the right hand side of (24), while the two upward sloping (dashed) lines correspond to the left hand side of (24), for the two different growth rates $\gamma_H$ and $\gamma_L$, assuming no sunspot shock.\footnote{The fact that $\frac{q_{t+1}^{\gamma_H}(1+q_{t+1})}{\gamma_H} < \frac{q_{t+1}^{\gamma_L}(1+q_{t+1})}{\gamma_L}$ is due to the assumptions of $\sigma > 2$ and $\alpha > \frac{1}{\sigma-1}$ (from which, $\chi > 0$) together with $\gamma_H > \gamma_L$.} Suppose that $\gamma_t = \gamma_L$. As the picture shows, for a given $q_t < q^L$, $q_{t+1}$ will be higher (respectively, lower) the higher (respectively, the lower) the realization of the growth rate $\gamma_{t+1}$. In particular, if $q_t$ is sufficiently close to $q^c$ - at which the right hand side of (23) peaks - a high $\gamma_{t+1}$ will make $q_{t+1}$ very close to $\phi$, increasing the chances that, despite a likely new shock to $\gamma_t$, $q_{t+2}$ will take a value very close to zero - i.e. making it more likely for the price-dividend ratio $q_{t+2}^{-1}$ to jump to a very high value. In our simple graphical example, for $\gamma_t = \gamma_L$ and an initial $q_0 \in (0, q^L)$, the dividend-price ratio moves closer to the (stochastic) steady state $q^L$. If $\gamma_t$ remained equal to $\gamma_L$, $q_t$ would keep fluctuating around $q^L$, and eventually it would either converge to it or to a period-$n$ cycle or be chaotic. Now suppose that after three periods, i.e. when $q_t = q_3$, the growth rate jumps to $\gamma_H$ for one period, and then goes back to $\gamma_L$. This will imply a jump to $q_4$ close to $\phi$, and, from there, a sudden drop to the very low $q_5$. Along this path, we would observe a price-dividend ratio $q_{t+1}^{-1}$ that, after some cyclical fluctuations around its long-run mean, suddenly booms. With $\gamma_t$ back to $\gamma_L$, $q_t$ will start reverting back to $q^L$. The price-dividend ratio $q_{t+1}^{-1}$ would then show another period of bounded fluctuations, until $\gamma_H$ gets realized. If, when this positive shock takes place, $q_t$ is in a neighborhood of $q^c$ a new boom-bust cycle occurs and the process repars itself.

The same mechanism works if we introduce the expectational error $\nu_{t+1}$. For instance, suppose that $\varepsilon_t = 0$ and hence $\gamma_t = \bar{\gamma}$ at all times, and that $\nu_{t+1}$ switches stochastically between a high value, $\nu_H$, and a low value, $\nu_L$. Similarly to the case of a switching growth rate, this time-variation would change the left-hand-side of (24) without any impact on its right-hand-side. This is displayed in panel b). Similarly to the dynamic process described in panel a), the price-dividend ratio suddenly takes a very high value if the realization of $\nu_H$ occurs when $q_t$ is somewhere in the neighborhood of $q^c$.\footnote{More generally, we can establish the following equivalence: for $\rho = 0$ (iid dividends growth) the equilibrium dynamics under the fundamental and the sunspot shock are identical provided that $1 + \nu_t = \exp((\sigma - 1)(\bar{x} + \varepsilon_t))$ holds at any $t$.}

### 5 Stabilization Policy

In our economy, non-fundamental cyclical and chaotic fluctuations arise because of the the agents seeking economic status. As shown, absent the spirit of capitalism, the economy would feature a unique equilibrium along which the growth-adjusted stock prices are constant. The objective of this section is to assess whether there exist simple fiscal instruments that can either restore the equilibrium uniqueness result or at least eliminate the cyclical/chaotic dynamics. An appropriate welfare analysis would try to conclude whether...
Figure 11: The equilibrium effects of fundamental and non-fundamental shocks. Panel a) displays the dynamic equilibrium effects of a time-varying dividends growth rate (e.g. a switch between $\gamma_L$ and $\gamma_H > \gamma_L$), for a generic initial condition $q_0 < q^*$. The figure plots the left and the right hand side of equation (24), for the two alternative cases $\gamma_L$ and $\gamma_H$ occurring at $t = 3$. Panel b) is identical except that the time variation concerns the sunspot shock $\nu_{t+1}$. 
these instruments - that is, taxes - are desirable from an optimal policy point of view.

For the time being, we are going to consider two simple fiscal policies: taxing dividends and taxing capital gains. This approach relates us to the vast and ongoing debate on whether the government should or should not implement tax cuts on these two sources of financial returns. A recent contribution by Anagnostopoulos et al. (2010) studies the consequences of cutting either one or both taxes for what concerns capital accumulation and social welfare, in an economy characterized by agents heterogeneity and uninsurable risk, i.e. incomplete markets. They conclude that both types of tax cuts - but in particular a dividend tax cut - are welfare reducing since, by raising the value of the current capital in use (though a positive asset price effect) they lower the people’s incentive to make further investments, being them already better off with respect to the pre-cut situation. Santoro and Wei (2009) instead focus on the asset price implications of taxing dividends versus taxing corporate profits, that is, between taxing at the household or at the firm level. Since dividends taxation is non-distortionary, a related tax cut does not have any impact on asset returns. On the other hand, taxing corporate profits increases aggregate uncertainty, and hence tends to raise the equity risk premium.

We are going to study the impact of dividends and capital gains taxation from the following perspective: for given \( \beta, \sigma \) and \( \alpha \) satisfying the conditions spelled in either Proposition 5 or 6, would a tax (on dividends or capital gains) lower or increase the minimum growth rate above which cycles and chaos occur?

Assume that the fiscal authority can tax dividends at a constant rate \( \tau^d \in (0, 1) \) and capital gains at a constant rate \( \tau^p \in (0, 1) \), and that tax revenues are fully rebated to the households through lump sum transfers \( T_t \). The representative agent’s budget constraint becomes:

\[
c_t + p_t a_t = [p_t + (1 - \tau^d) d_t] a_{t-1} - \tau^p (p_t - p_{t-1}) a_{t-1} + T_t
\]

Similarly to Anagnostopoulos et al. (2010), in order to simplify the analysis, we are assuming that capital gains taxes are paid on an accrual basis and that capital losses are subsidized at the same rate.\(^{14} \) The government budget constraint is a simple balanced-budget one:

\[
\tau^d d_t a_{t-1} + \tau^p (p_t - p_{t-1}) a_{t-1} = T_t
\]

Obviously, our agents are atomistic and hence do not internalize that taxes will be fully rebated to them.

Solving the representative agent’s optimization problem and imposing equilibrium conditions, we obtain the following intertemporal condition:

\[
\beta d_t^{(1-\alpha)(1-\sigma)-1} p_t^{(1-\sigma)\alpha} [(1 - \tau^p) p_{t+1} + (1 - \tau^d) d_{t+1} + \tau^p p_t] = d_t^{(1-\alpha)(1-\sigma)-1} p_t^{(1-\sigma)\alpha} \left(1 - \frac{\alpha}{1 - \alpha} p_t\right)
\]

\(^{14}\)This assumption is not innocuous, as, in reality, capital gains are taxed only if realized. See Gavin et al. (2007) for an example on how to model realization-based capital gains taxes.
Simple algebra shows that the equilibrium dynamics of the dividend-price ratio \( q_t \) are described by the following dynamic equation:

\[
q_{t+1}^\chi \left( 1 - \tau^p + (1 - \tau^d) q_{t+1} + \frac{\tau^p q_{t+1}}{\gamma} q_t \right) = \frac{\gamma^{(\sigma-1)}}{\beta} q_t^\chi \left( 1 - \frac{q_t}{\phi} \right)
\]

where \( \chi \) and \( \phi \) are as in (8)-(9).

### 5.1 Taxing Dividends

We are going to consider dividends and capital gains taxation separately. Starting with the former, let \( \tau^p = 0 \). Notice that the dynamic equation (26) reduces to:

\[
q_{t+1}^\chi \left( 1 + (1 - \tau^d) q_{t+1} \right) = \frac{\gamma^{(\sigma-1)}}{\beta} q_t^\chi \left( 1 - \frac{q_t}{\phi} \right)
\]

This is exactly equivalent to (7) except for the coefficient \( (1 - \tau^d) \) multiplying \( q_{t+1} \) on the left hand side.

One should immediately realize that the introduction of dividends taxation does not affect the main result stated in Proposition 1: that is, absent the spirit of capitalism, i.e. \( \alpha = 0 \), the unique equilibrium is still the steady state, which would be \( q^* = \frac{2^{(1-\sigma)-1}}{1-\tau^d} \). Taxing dividends would simply increase the steady state dividend-price ratio.

Taxing dividends does not qualitatively affect our results for the low risk aversion economy as well. For \( \sigma \in (0, 2) \), the economy still displays the steady state as a viable equilibrium - although it is now equal to \( q^*_t \equiv \phi \frac{s^{(\sigma-1)}_{1-\tau^d}}{(1-\tau^d)\phi + \frac{s^{(\sigma-1)}_{1-\tau^d}}{1-\tau^d}} > q^* \) - together with a continuum of dynamic equilibria each indexed by an initial condition \( q_t \in (0, q^*_t) \).

Now, let’s consider the case of \( \sigma > 2 \) and \( \alpha > \frac{1}{\sigma-1} \) of Proposition 5. Following similar steps, one can show that there exist a \( \gamma^*_\tau \) and a \( \tau^*_\tau \) such that period-2 cycles occur for \( \gamma \in (\gamma^*_\tau, \tau^*_\tau) \), and similarly that there exists a \( \gamma^*_\tau \) above which we observe period-3 cycles and chaotic dynamics, where the subscript \( \tau \) has to do with the fact that those thresholds depend now on \( \tau^d \). We state these results formally, as well as the fact that an increase in the dividends tax rate \( \tau^d \) lower the bifurcation thresholds, in the next Proposition.

**Proposition 7** Suppose that \( \sigma > 2 \) and that \( \alpha > \frac{1}{\sigma-1} \), and define \( \tau^*_\tau \equiv \left[ \beta \left( 1 + \phi^\tau \right) \left( 1 + \chi^\tau \right) \right]^{1/\chi} \) for \( \phi^\tau \equiv (1 - \tau^d) \phi \). Then, there exist a threshold value \( \gamma^*_\tau \in (1, \tau^*_\tau) \) such that period-2 cycles occur for \( \gamma \in (\gamma^*_\tau, \tau^*_\tau) \). Moreover, \( \frac{\partial \gamma^*_\tau}{\partial \alpha} < 0 \) and \( \frac{\partial \gamma^*_\tau}{\partial \tau^d} < 0 \).

**Proof.** The proof of the existence of period-2 cycles is identical to the one of Proposition 5. Let \( \phi^\tau \equiv (1 - \tau^d) \phi \). Following similar steps one can show that period-2 cycles occur for \( \gamma \in (\gamma^*_\tau, \tau^*_\tau) \) where \( \gamma^*_\tau \equiv \left[ \beta \left( 1 + \delta^*_\tau \right) \right]^{1/\chi} \),

\[
\delta^*_\tau \equiv \frac{- [1 - \phi^\tau - 2\chi (1 + \phi^\tau)] + \sqrt{(1 - \phi^\tau)^2 + 4\chi (1 + \phi^\tau)^2 (1 + \chi)}}{2}
\]
and $\gamma^c < \bar{\gamma}$. Clearly, since $\sigma > 2$ and $\alpha > \frac{1}{\sigma-1}$ (hence $\chi > 0$), $\bar{\gamma}$ is strictly increasing in $\phi^*$, and as a consequence strictly decreasing in $\tau^d$: that is, $\frac{\partial \bar{\gamma}}{\partial \tau^d} < 0$. Now consider $\gamma^c$. This is strictly increasing in $\delta^c$. So we simply need to show that the latter is strictly increasing in $\phi^*$. Taking first order derivative with respect to $d$ we have that

$$
\frac{\partial \gamma^c}{\partial \tau^d} > 0
$$

if and only if

$$
(1 + 2\chi) \sqrt{(1 - \phi^*)^2 + 4\chi (1 + \phi^*)^2 (1 + \chi) + 4\chi (1 + \phi^*) (1 + \chi) - (1 - \phi^*) > 0}
$$

(28)

Let the left hand side of this inequality be $LHS$, and notice that

$$
LHS > (1 + 2\chi) (1 - \phi^*) + 4\chi (1 + \phi^*) (1 + \chi) - (1 - \phi^*)
$$

$$
= 2\chi (1 - \phi^*) + 4\chi (1 + \phi^*) (1 + \chi)
$$

$$
= 4\chi^2 (1 + \phi^*) + 2\chi (3 + \phi^*) > 0
$$

Hence the inequality (28) holds and we have $\frac{\partial \gamma^c}{\partial \tau^d} < 0$ as well.

Figure 12 presents the orbit bifurcation diagram with respect to the dividends tax rate $\tau^d$ for the case of $\beta = 0.92$, $\sigma = 2.5$, and $\alpha = 0.69$ analyzed in Section ???. In this experiment, we fix the growth rate $\gamma = 1.03$ - for which the steady state is the unique limiting point when $\tau^d = 0$ - and let $\tau^d$ vary within the range $[0, 0.9]$. The numerical analysis clearly shows that as the dividends tax rate becomes slightly higher than 20% period-2 cycles appear, and that, if increased further, we would observed higher order cycles and eventually chaotic dynamics.

### 5.2 Taxing Capital Gains

Next, we consider the case of $\tau^p \in (0, 1)$ while setting $\tau^d = 0$. The dynamic equation (26) reduces to:

$$
q_{t+1}^\chi \left(1 - \tau^p + q_{t+1} + \frac{\tau^p q_{t+1}}{\gamma q_t} \right) = \frac{\gamma^{(\sigma-1)}}{\beta} q_t^\chi \left(1 - \frac{q_t}{\phi^*} \right)
$$

(29)

This case clearly presents some additional analytical complexity with respect to the tax on dividends. While the right hand side is unchanged, the left hand side now depends both on $q_{t+1}$ and $q_t$, making the characterization of the map between current and future dividend-price ratios more complicated. Although a complete analytical characterization of our results is not attainable, we can prove that, as long as the dividends growth rate is not too high, there exists a map $F : (0, 1) \rightarrow (0, 1)$ such that $q_{t+1} = F (q_t)$ is the solution to (29) for any $q_t \in (0, 1)$. In order to make our results comparable to what obtained under dividends taxation, we restrict to the case of $\sigma > 2$ and $\alpha > \alpha^m$.

**Proposition 8** Assume that there exists a positive tax on capital gains, $\tau^p \in (0, 1)$, and that $\sigma > 2$ together with $\alpha > \alpha^m$. Moreover, define $\varepsilon \equiv 1 - \left[ \frac{\tau^p}{\gamma} + (1 - \tau^p) \right] \in (0, 1)$ and

$$
\bar{\tau} = \left[ \beta (1 - \varepsilon + \phi) (1 + \chi)^{1+\varepsilon} \right]^{\frac{1}{\sigma-1}} > 1
$$

30
Then, if $\gamma < \gamma^*$, there exists a mapping $F_{\tau} : (0, \phi) \rightarrow (0, \phi)$ such that $q_{t+1} = F_{\tau} (q_t)$ solves (29) for any $q_t \in (0, \phi)$, and $F$ has the following properties:

1. $\lim_{q_{t+1} \to 0} F_{\tau} (q_t) = \lim_{q_{t+1} \to \phi} F_{\tau} (q_t) = 0$ and $\lim_{q_t \to 0} F'_{\tau} (0) > 1;$

2. there exists a unique steady state solution $q^*_t = \frac{1}{\phi + \frac{1}{\sigma} + \frac{1}{\beta}} > 0$ where $q^*_t = F_{\tau} (q^*_t)$. 

Given these properties, a sufficient condition for the existence of period-2 cycles is $F'_{\tau} (q^*_t) < -1$.

**Proof.** First of all, we are going to show that, under the condition $\gamma < \gamma^*$, for any $q_t \in (0, \phi)$ there exists a unique $q_{t+1} \in (0, \phi)$ solving the dynamic equation (29). That is, we have a well-defined mapping $F_{\tau} : (0, \phi) \rightarrow (0, \phi)$.

Let $K (q_{t+1}, q_t) \equiv q_{t+1} \chi \left( 1 - \tau^p + q_{t+1} + \frac{\tau^p}{q_t} \right)$ and $H (q_t) \equiv \frac{\tau^{(-1)} (q_t - \frac{q_t}{\phi})}{\phi + \frac{1}{\sigma} + \frac{1}{\beta}}$, where, for $\sigma > 2$ and $\alpha > \frac{1}{\sigma - 1}$, we have that $\chi > 0$. For a given $q_t \in (0, \phi)$, consider $K (q_{t+1}, q_t)$ as a function of $q_{t+1}$. We can easily see that $\lim_{q_{t+1} \rightarrow 0} K (q_{t+1}, q_t) = 0$, $\lim_{q_{t+1} \rightarrow \phi} K (q_{t+1}, q_t) = 1 - \tau^p + \phi + \frac{\tau^p}{q_t}$, and $\frac{\partial K (q_{t+1}, q_t)}{\partial q_{t+1}} > 0$. Clearly, for any given $q_t \in (0, \phi)$, a unique solution $q_{t+1} \in (0, \phi)$ to $K (q_{t+1}, q_t) = H (q_t)$ exists if and only if $1 - \tau^p + \phi + \frac{\tau^p}{q_t} > H (q_t)$. Let $J (q_t) \equiv 1 - \tau^p + \phi + \frac{\tau^p}{q_t}$ denote the left hand side of this inequality. From the proof of Proposition 4, we know that $\lim_{q_{t+1} \rightarrow 0} K (q_{t+1}, q_t) = \lim_{q_{t+1} \rightarrow \phi} K (q_{t+1}, q_t) = 0$ and that $H (q_t)$ is single peaked at $q^*_t \equiv \frac{\phi}{1 + \gamma}$. Since $J (q_t)$ is strictly decreasing in $q_t$ with $\lim_{q_{t+1} \rightarrow 0} J (q_t) = 1 - \tau^p + \phi + \frac{\tau^p}{q_t}$, a sufficient condition
for $1 - \tau^p + \phi + \frac{\tau^p}{q_t} > H(q_t)$ to hold for any $q_t \in (0, \phi)$ - and hence for a solution $q_{t+1}$ to (29) to exist for any $q_t \in (0, \phi)$ - is $1 - \tau^p + \phi + \frac{\tau^p}{q_t} \geq H(q_t')$. After simple algebra, the latter can be written as $\gamma < \gamma^*$ where $\gamma^* \equiv \left[\beta (1 - \varepsilon + \phi) \frac{(1 + \chi)^{1+\varepsilon}}{\chi}\right]^{\frac{1}{p+1}}$. Under this condition, there exists a mapping $F_r : (0, \phi) \rightarrow (0, \phi)$ such that $q_{t+1} = F_r(q_t)$ solves (29) for any $q_t \in (0, \phi)$.

The property spelled in 1. follows from the fact that $\lim_{q_t \rightarrow 0} H(q_t) = \lim_{q_t \rightarrow 0} H(q_t) = 0$ together with $\lim_{q_{t+1} \rightarrow 0} K(q_{t+1}, q_t) = 0$. The slope of $F_r(q_t)$ is obtained by the Implicit Function Theorem, whereas:

$$F'_r(q_t) = \left(\frac{q_{t+1}}{q_t}\right)^2 \frac{\tau^p}{\tau} + \frac{\tau^p}{q_t} \frac{1 + \chi}{\chi(1 - \tau^p) + \chi q_{t+1}} + \frac{\tau^p q_{t+1}}{q_t} + q_{t+1} \left(1 + \frac{\tau^p}{\gamma q_t}\right)$$

for $q_{t+1} = F_r(q_t)$ (30) It is straightforward to notice that $F'_r(q_t) > 0$ for sure if $q_t \leq \frac{\phi \chi}{1 + \chi}$. Although this property together with the fact that $\lim_{q_t \rightarrow 0} F_r(q_t) = \lim_{q_t \rightarrow \phi} F_r(q_t) = 0$ implies that there must be a value $\tilde{q} \in (0, \phi)$ where $F_r'(\tilde{q}) = 0$, the map $F$ is not necessarily single-peaked. Moreover, from $\lim_{q_t \rightarrow 0} F_r(q_t) = 0$, by simple algebra, one obtains that $\lim_{q_t \rightarrow 0} F'_r(q_t) = \frac{\tau^p + \frac{(1+\varepsilon)}{\beta} \chi}{\chi(1 - \tau^p) + \chi q_{t+1} + \frac{\tau^p q_{t+1}}{q_t} + q_{t+1} (1 + \frac{\tau^p}{\gamma q_t})}$. The latter is always bigger than $1$ since $\frac{(1+\varepsilon)}{\beta} > 1$ and $\frac{\tau^p (1 - \tau^p)}{\gamma} \in (0, 1)$. Setting $q_{t+1} = q_t = q$ in (29), we obtain the unique steady state solution $q^* = \phi \frac{\frac{(1+\varepsilon)}{\beta} - 1 + \varepsilon}{\phi + \frac{(1+\varepsilon)}{\beta}} > 0$, which proves the statement in 2.

Given these features, similarly to Proposition 5, a sufficient condition for the existence of period-2 cycles is that $F'_r(q^*) < -1$, i.e. the steady state has to be dynamically unstable. ■

In Figure 13, the orbit bifurcation diagram is with respect to the capital gains tax rate $\tau^p$. We have fixed $\sigma = 2.5$ and $\alpha = 0.69$, such that, absent the tax, we are in the case of Proposition 5. However, we depart from the benchmark parametrization by assuming a rather low discount rate, $\beta = 0.8$, while setting $\gamma = 1.03$. The reason is the following: in our framework, taxing capital gains actually reduces the possibility of cycles and chaos. Therefore, we want to consider a case where the economy displays complex fluctuations for $\tau^p = 0$ and assess to what extent setting a positive $\tau^p$ can rule out those dynamics. Based on the results reported in Figure 3 and 5 we have two options: either we pick a high growth rate (around 1.15) while keeping $\beta = 0.92$ (see Figure 1) or we pick a lower discount rate (below 0.8) while setting $\gamma$ reasonably low. Without loss of generality, we have decided to go for the lower discount rate case.

As the figure shows, it is enough to increase the tax rate $\tau^p$ from zero to about 4% to eliminate all cyclical fluctuations and re-establish the steady state as the unique limit point.

In light of these results, we assess whether the stabilizing (respectively, destabilizing) consequences of taxing capital gains (respectively, dividends) hold in the stochastic environment. Figure ?? displays the simulation results for the case of a positive tax on dividends (panels a)-c)) and a positive tax on capital gains (panel d)-f)) with $\sigma = 2.5$, $\alpha = 0.69$ and $\beta = 0.92$. The stochastic process for dividends growth features $\rho = 0.5$, $E(\gamma_t) = 1.05$ and $\sigma_\varepsilon = 0.01$. It clearly emerges that taxing dividends exacerbates fluctuations also
Figure 13: **Orbit bifurcation diagram with respect to the capital gains tax** $\tau^p$. The figure displays the orbit bifurcation diagram with respect to the capital gains tax $\tau^p$. Calibration: $\sigma = 2.5$, $\alpha = 0.69$, $\beta = 0.92$ and $\gamma = 1.03$.

in the deterministic case. On the other hand, a minimal tax on capital gains - as low as 10% - is sufficient to virtually eliminate all boom-bust cycles.

What if dividends and capital gains are taxed simultaneously? Would the stabilizing properties of $\tau^p$ or the destabilizing properties of $\tau^d$ prevail? The results of this experiment are displayed in Figure ???. Moving from left to right along the panels, one can see that taxing capital gains at a mere 5% is already sufficient to dramatically reduce the volatility of the price-dividend ratio, independently from the value assigned to $\tau^d$.

## 6 Some Extensions

### 6.1 Individual versus Aggregate Wealth

We now assume that the agents' status depends both on their individual wealth (positively) as well as the average wealth in the economy (negatively). This captures the possibility that a single agent might perceive his/her status to worsen if he/she witnesses an increase in others' wealth. This is accomplished by dropping Assumption 2 and letting $\mu > 0$ in (1).\(^{15}\) However, we do not restrict to the case of $\mu = 1$, for which status is measured by relative wealth. By letting $\mu \in (0,1)$ we are able to capture the possibility that the agents either a) care about their own wealth more in absolute than in relative terms, or b) might have a subjective

\(^{15}\)While doing this, we keep the assumption of unitary elasticity of substitution between consumption and status in utility, which, therefore, maintains the Cobb-Douglas form.
Figure 14: Simulated price-dividend ratio and taxation. In panel a)-c), capital gains are not taxed ($\tau^p = 0$), while dividends are taxed at 0, 10 or 30%. In panel d)-f), dividends are not taxed ($\tau^d = 0$), while capital gains are taxed at 0, 10 or 30%. Calibration: $\sigma = 2.5$, $\alpha = 0.69$ and $\beta = 0.92$. The stochastic process for the growth rate features: $\rho = 0.5$, $E(\gamma_t) = 1.05$ and $\sigma_{\epsilon} = 0.01$. The sunspot is iid with $\nu_t \in (-0.01, 0.01)$. 
Figure 15: **Simulated price-dividend ratio and combined taxation.** Panels a) to c) show the stabilizing effects of mild capital gains taxation ($\tau^p$ goes from 0 to 10%) while keeping $\tau^d = 0$. Panels a), d) and g) show the de-stabilizing effects of dividends taxation ($\tau^d$ goes from 0 to 30%) while keeping $\tau^p = 0$. The remaining panels consider the combination of dividends and capital gains taxation. Calibration: $\sigma = 2.5$, $\alpha = 0.69$ and $\beta = 0.92$. The stochastic process for the growth rate features: $\rho = 0.5$, $E(\gamma_t) = 1.05$ and $\sigma_\epsilon = 0.01$. The sunspot is iid with $\nu_t \in (-0.01, 0.01)$. 


and possible imprecise perception of others’ wealth.

Using the definition of $q_t$, Assumption 3 and simple algebra, we obtain the following dynamic equation:

$$q_{t+1} = \frac{\gamma^{\chi} \left[ (\sigma-1)(1-\alpha + \alpha(1-\mu)) \right] \beta}{\phi^{\chi}} \left( 1 - \frac{q_t}{\phi} \right)$$

(31)

where now $\chi \equiv (\sigma - 1)\alpha (1 - \mu) - 1$, while $\phi \equiv \frac{1 - \alpha}{\alpha} > 0$ as before. From the previous analysis, we know that cyclical and chaotic equilibria are more prominent for the case of $\chi > 0$, which requires $\sigma > 2$ and $\alpha > \frac{1}{\sigma - 1} \equiv \alpha^m$. With $\mu > 0$, these conditions are stricter, as $\chi > 0$ now requires $\sigma > 1 + \frac{1}{1-\mu} > 2$ and $\alpha > \frac{1}{(\sigma-1)(1-\mu)} \equiv \alpha^m$. Moreover, for the extreme case of $\mu = 1$ (status depends on relative wealth only), we have that $\chi = -1$, a case for which, similarly to what obtained in Proposition 3, cycles and chaos can not occur.\(^\text{16}\)

### 6.2 Endogenous Production and Dividends

In this section we endogenize dividends by introducing labour as an input to production. Under this new set-up the representative agent solves the following problem:

\(^\text{16}\)It is possible to show, both analytically and numerically, that, conditional on $\sigma > 1 + \frac{1}{1-\mu}$ and $\alpha > \frac{1}{(\sigma-1)(1-\mu)}$, the bifurcation value of $\gamma$ above which cyclical dynamic occur is increasing in $\mu$. A more detailed analysis is available from the author upon request.
max \sum_{t=0}^{\infty} \beta^t \left[ \frac{(c_t^{1-\alpha} s_t^\alpha)^{1-\sigma}}{1-\sigma} - A h_t^{1+\varphi} \right]

subject to

c_t + p_t a_t = (p_t + d_t) a_{t-1} + n_t h_t

where \( n_t \) is the hourly wage. Production takes place in a perfectly competitive sector: a representative firm hires labour from the household sector to produce the final consumption good under decreasing returns to scale. The firm’s problem is:

\[ \max z_t h_t^\theta - n_t h_t \]

where \( \theta \in (0,1) \) and \( z_t = \gamma t z \) is a deterministic productivity process. The firm’s profits are then distributed to the agents in the form of dividends, which, in equilibrium are equal to a fraction \( (1-\theta) \) of output: \( d_t = z_t (1-\theta) h_t^\theta \). After imposing the market clearing condition \( c_t = z_t h_t^\theta \) and extensive algebra, the equilibrium dynamics of the dividend-price ratio \( q_t \) reduce to the following non-linear difference equation:

\[ q_{t+1}^\chi (1 + q_{t+1}) = \frac{\gamma \omega}{\beta} q_t^\chi \left( 1 - \frac{q_t}{\phi} \right) \]

where now \( \phi \equiv (1-\theta) \frac{1-\alpha}{\alpha} > 0 \), \( \chi \equiv \frac{(1+\varphi)(\sigma-1)\alpha-1-\varphi(\sigma-1)\theta}{(1+\varphi)(\sigma-1)+\varphi(\sigma-1)\theta} \) and \( \omega \equiv \frac{(1+\varphi)(\sigma-1)}{(1+\varphi)+\varphi(\sigma-1)} \). Notice that for \( \theta = 0 \) we are back to the case of endogenous dividends, as \( d_t = \gamma t z \), and the dynamic equation in (32) becomes identical to (7) as \( \chi = (\sigma - 1) \alpha - 1 \) and \( \omega = \sigma - 1 \).

We focus on the case of \( \chi > 0 \) which has been extensively analyzed in Section 3.2. Under its new definition, \( \chi > 0 \) for \( \sigma > 1 + \frac{1+\varphi}{1+\varphi-\theta} > 2 \) and \( \alpha > \frac{1}{\sigma-1} + \frac{\theta}{1+\varphi} \). Comparing this with the conditions in Propositions 4-5, it appears that the existence of cyclical and chaotic dynamics in an economy with endogenous dividends requires a higher CRRA and a higher degree of SOC with respect to an economy where the paths of dividends is exogenous. Nevertheless, provided that both conditions are satisfied, it can be shown that period-2 and higher order cycles, as well as chaos, occur now at lower \( \gamma \) values.\(^1\) This result can be proved analytically following similar steps to those highlighted in Propositions 4-6 However, to avoid lengthy analytical proofs, we summarize it by presenting the orbit-bifurcation diagram with respect to the growth rate \( \gamma \), keeping all structural parameters as in our benchmark calibration of Section ??.

\(^1\) Although \( \gamma \) has different meanings in the two economies considered - i.e. it is the constant dividends growth rate in the benchmark case, while it is the growth rate of a productivity shock here - in both cases there exists a unique balanced growth path where all endogenous variables grow at the same rate \( \gamma \).
Figure 17: Orbit bifurcation diagram with endogenous dividends. The figures displays the orbit bifurcation diagram with respect to the growth rate $\gamma$ when production (and therefore dividends) are endogenous.

7 Conclusions

In the standard Lucas’ tree asset pricing model, under rational expectations, stock price volatility fully reflects the underlying fundamentals. For instance, the market price of a stock producing constant dividends would be constant as well. Though this is a rather extreme and unrealistic case, it clearly points out the inadequancy of such paradigm to account for the volatile stock price dynamics observed in the data.

Rather than abandoning the rational expectations hypothesis and exploring other mechanisms of expectations formation - as in Adam et al. (2007) and Lansing (2010) - we explore the asset pricing implication of Max Weber’s (1958) spirit of capitalism. In our model, economic agents care about their social status, which we have assumed to be related to own financial wealth, similarly to Bakshi and Chen (1996) and, more recently, Kamihigashi (2007). While the latter shows that under this hypothesis we can not rule out stock price bubbles on the basis of "non-optimality", we find that complex stock price dynamics can occur if individual preferences are non-separable between consumption and status.

For a deterministic version of our economy, we show that for a wide range of realistic values of the CRRA coefficient, the combination of a sufficient spirit of capitalism and of sufficient dividends growth induces cyclical as well as chaotic stock price dynamics. Period-2 cycles can also occur at low degrees of SOC, but this seems to be a much more restricted possibility. Moreover, in this case, the equilibrium dynamics are well-defined only in a backward sense. This requires some additional analysis along the lines of Medio and Raines (2007) as well as Kennedy and Stockman (2008).
Once fundamental uncertainty is introduced - in the form of a shock to dividends growth - the model is capable of generating recurrent boom-bust cycles which are completely endogenous and resemble the stock price bubble episodes observed in the real world. Interestingly, these volatile dynamics are obtained without necessarily assuming the existence of sizable (sunspot-driven) expectational errors. We assess how the existence of these boom-bust phases depends on the volatility of the fundamental shock, its persistence, as well as on the long-run mean of the dividends growth rate. While for the deterministic case, cycles and chaos require a moderate degree of dividends growth, under uncertainty, recurrent bubbles and crashes can occur at rather small growth rates.

With complex dynamics and endogenous bubbles-crashes arising because of structural reasons, one might wonder whether there exist appropriate policies to eliminate the the excessive stock price volatility. In this respect to explore the consequences of taxing either dividends or capital gains. Interestingly, we find that a minimal tax on capital gains can eliminate both cycles and chaos, while dividends taxes have the opposite effects. Other things equal, an increase in the dividends tax rates lowers the minimum growth rate above which period-2 cycles occur. That is, under dividends taxation, cycles and chaos can occur even in zero-growth economies.

Our analysis is still at a preliminary stage and can indeed improved/extended in several directions. The first one is related to preferences. So far, for simplicity, we have assumed a Cobb-Douglas utility function and that agents care about their own status, without any reference to some aggregate level. We conjecture that, while our results would still remain under a more general CES utility function, the room for cycles and chaos would be smaller if we replaced the individual status with some relative measure.

Second, we have intentionally restricted our analysis to a Lucas’ tree environment, whereby dividends are completely exogenous. It would be interesting to assess the role of the spirit of capitalism under non-separable preference in a neo-classical growth model. Kamihigashi (2007) has made some interesting steps in this direction.

We are currently working on these extensions.

References
