Solutions and Phase Portraits of Endogenous Growth Models with Optimal Saving

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0.1 Introduction

The saving rate mostly varies over time, as it will be endogenously determined by infinitely lived consumers maximizing total lifetime utility.

0.2 Structure of Economic Dynamics with Optimal Saving

Let us consider infinitely lived consumers maximizing total lifetime utility. The representative consumer is assumed to have a time additive intertemporal utility function

\[ U = \int_{0}^{\infty} u[c(t)]e^{-\rho t} dt, \quad c = C/L \]

(1)

where the decision variable, \( c \), is per capita consumption and \( \rho \) is the constant subjective rate of time preference (discount rate). The momentary utility (felicity) function, \( u(c) \), is assumed to be concave, increasing, and to satisfy the Inada conditions

\[ \lim_{c \to 0} u'(c) = \infty, \quad \lim_{c \to \infty} u'(c) = 0. \]

(2)

With \( \rho > 0 \), total utility, \( U \) is bounded, if \( u(c) \) with (2) is also bounded over time.

With exogenous labor growth \( L(t) = L_0 e^{nt} \),

We shall consider optimal saving in an economy with exogenous population(labor) growth. A social planner may maximize the objective function

\[ V = \int_{0}^{\infty} u[c(t)]L(t)e^{-\rho t} dt = L_0 \int_{0}^{\infty} u[c(t)]e^{-(\rho - n)t} dt \]

(3)

Barro and Sala-i-Martin (1995),

The technology, \( F(L, K) \), is described by a smooth concave homogeneous production function with constant returns to scale in labor and capital,

\[ Y = F(L, K) = Lf(k) \equiv Ly, L \neq 0; \quad F(0, 0) = 0 \]

(4)

where the function \( f(k) \), is strictly concave and monotonically increasing in the capital-labor ratio \( k \in [0, \infty] \), i.e., \( f(k) \) has the properties

\[ \forall k > 0: \quad f'(k) > 0, \quad f''(k) < 0 \]

(5)

\[ \lim_{k \to 0} f'(k) \equiv b \leq \infty, \quad \lim_{k \to \infty} f'(k) \equiv b' \geq 0, \quad f'(k) \in [b, b'] \]

(6)

For the one-sector model, the capital accumulation is given by, cf. (1),

\[ \dot{K} = Y - cL - \delta K = L[f(k) - c - \delta]. \]

(7)

\[ \dot{k} = f(k) - c - (n + \delta)k. \]

(8)

Thus the Ramsey optimization problem is

\[ \max_{c(t)} V = \max_{c(t)} \int_{0}^{\infty} u[c(t)]e^{-(\rho - n)t} dt \]

(9)

s.t. \( \dot{k} = f(k) - c - (n + \delta)k, \quad c \geq 0, \)

(10)
which is equivalent to maximizing the current value Hamiltonian function

$$H = u[c(t)] + \lambda(t) [f(k) - c - (n + \delta)k],$$

with a costate variable, $\lambda(t)$, and the transversality conditions:

$$k(0) = k_0, \quad \lim_{t \to \infty} \lambda(t)k(t)e^{-(\rho-n)t} = 0.$$  

(12)

The first order condition gives, cf. (11)

$$\frac{\partial H}{\partial c} = u'(c) - \lambda(t) = 0, \quad (13)$$

By time derivation of (13) and next inserting the first order condition (13) into (14), we obtain the differential equation for per capita consumption, $c$, as

$$\dot{c} = -\frac{u'(c)}{u''(c)} [f'(k) - (\delta + \rho)] \equiv \psi(c) c [f'(k) - (\delta + \rho)] \quad (15)$$

$$\dot{c}/c = \hat{c} = \psi(c) \frac{c}{u''(c)} [f'(k) - (\delta + \rho)] \quad (16)$$

where, $\psi(c) = -u'(c)/[u''(c)c]$, is the intertemporal elasticity of substitution. By (13) the transversality condition (12) becomes

$$u'(c)k(t)e^{-(\rho-n)t} \to 0 \text{ as } t \to \infty.$$  

(17)

The differential equations (8), (15), define a dynamic system in $k$ and $c$ with the governing function, $h(k, c)$, and $\eta(k, c)$ on the closed first quadrant, $\mathbb{R}^2_+$:

$$\dot{k} = h(k, c) = f(k) - (n + \delta)k - c, \quad (18)$$

$$\dot{c} = \eta(k, c) = \psi(c) c [f'(k) - (\delta + \rho)], \quad (19)$$

Remark. The sufficiency of (9)–(14) can be proved simply. It is observed that the objective function $u[c(t)]e^{-(\rho-n)t}$ is a concave function in $(k, c)$-space, cf. (2). Furthermore, it can be shown, by using (13) and (2), that $\lambda(t) = u'(c) > 0$, and that the function, $h(k, c)$, in (10) is also concave in $c, k$. Therefore the necessary conditions provided by the maximum principle are also sufficient for optimal solutions. △

0.3 Solutions and Phase Portraits of Endogenous Growth

When there are no critical points of (18-19), persistent per capita growth prevails as depicted in Figure 1.2. To characterize this situation, we give

**Theorem 1.** With optimal (Ramsey) saving, persistent (endogenous) per capita growth can be obtained if the concave per capita function $f(k)$ and
the intertemporal substitution elasticity $\psi(c)$ or the rate of time preference $\rho$ satisfies, respectively

$$\lim_{k \to \infty} f'(k) \equiv b > \delta + \rho, \quad (20)$$

$$\bar{\psi} = \sup_{c > 0} \psi(c) < \frac{b - (n + \delta)}{b - (\rho + \delta)} \Leftrightarrow \rho > (b - \delta)\left[\frac{\bar{\psi} - 1}{\psi}\right] + \frac{n}{\psi}, \quad (21)$$

The two conditions, (20), (21), ensure the existence – below the isocline $\dot{k} = h(k, c) = 0$ – of a separator, the particular orbit depicted in Figure 1.2:

$$\Gamma(t) \equiv [k^*(t), c^*(t)], \quad (22)$$

The existence of this separator, (22), is required for persistent (endogenous) per capita growth. With nondecreasing utility functions, $u(c)$, this separator in Figure 1.2 is also the unique optimal orbit (solution) satisfying (9)–(12).

**Proof.** We consider the dynamic system, (18), (19),

**Assumption A.** The per capita function $f(k)$, (5), has the continuity and differentiability properties as follows,

(i) $f(k) \in C^0([0, \infty]) \cap C^1([0, \infty])$, (ii) $f(0) \geq 0$. \quad (23)

It is further assumed that

(iii) $\forall k > 0 : f'(k) > \delta + \rho$. \quad (24)

For a concave function with $f(k) \to \infty$ as $k \to \infty$, (24) becomes, cf. (6),

(iv) $\lim_{k \to \infty} f'(k) \equiv b > \delta + \rho$. \quad (25)

It follows from (23)–(24) or (23) and (25) that the dynamic system (18)–(19) has no stationary solutions in $\mathbb{R}^2_+$ [except possibly for $(0, 0)$], and that the positive $k$-axis ($c = 0$) is a trajectory (orbit).

**Lemma 1A.** If there exists $d > 0$ and $k_0 > 0$ such that $\forall k \geq k_0, \forall c > 0$:

$$f(k)/k - (n + \delta) - \psi(c) [f'(k) - (\delta + \rho)] \geq d, \quad (26)$$
then there exists to the system (18)–(19) an orbit \( \Gamma(t) \equiv [k^*(t), c^*(t)], \ t \in \mathbb{R}, \) such that \( k^*(t) \to \infty, c^*(t) \to \infty, \) as \( t \to \infty \) – which separates the first quadrant into two regions I and II in figure 1.2. An orbit starting in the lower region I has the same behavior as \( \Gamma \) for \( t \to \infty, \) whereas an orbit starting in the upper region II eventually meets the \( c \)-axis, \( k = 0. \)

**Proof.** Consider the region \( W_\alpha = \{ (k, c) \mid 0 \leq c \leq \alpha k \land k \geq k_0 \} \), where \( \alpha \) is a positive constant chosen such that \( W_\alpha \) becomes positively invariant, cf. figure A.

![Figure 2: The positive invariant region, \( W_\alpha \), with endogenous (persistent) per capita growth](image)

Since the vector field (18)–(19) is directed inward on the line \( k = k_0 \), and since the positive \( k \)-axis is a trajectory, the region \( W_\alpha \) is positive invariant iff the vector field points inward on the line \( c = \alpha k, \ (k > k_0). \) Since the inward pointing normal to the line \( c = \alpha k \) is \( (\alpha, -1), \) we require

\[
\alpha h(k, \alpha k) - \eta(k, \alpha k) > 0, \quad \text{for } k \geq k_0. \tag{27}
\]

Inserting the expressions for \( h, \) (18), and \( \eta, \) (19), into (27), and simplifying, we find the requirement:

\[
\alpha < \frac{f(k)}{k} - (n + \delta) - \psi(\alpha k) \left[ f'(k) - (\delta + \rho) \right] \equiv R(k), \quad \text{for } k \geq k_0. \tag{28}
\]

A positively invariant region \( W_\alpha \) (with some \( \alpha > 0 \)) exists iff \( R(k) \) is bounded from below by a positive constant. By (26), we have for \( k \geq k_0 \)

\[
R(k) \geq d > 0. \tag{29}
\]

Choose \( \alpha \) to be any positive constant less than \( d. \) Then \( W_\alpha \) is positively invariant. For any orbit in the open first quadrant, \( \mathbb{R}^+_2, \) we have by (24) that \( \dot{c} > 0. \) Accordingly, it follows that any orbit starting in \( W_\alpha \) must satisfy \( k(t) \to \infty, c(t) \to \infty, \) as \( t \to \infty. \) Any orbit in \( \mathbb{R}^+_2 \) must either behave as just characterized (class I) or cross the \( \dot{k} = 0 \) nullcline (class II). In the latter case, the orbit will meet the \( c \)-axis eventually, since otherwise \( c(t) \to \infty \) and \( k(t) \to k_\varepsilon \) as \( t \to \infty, \) for some \( k_\varepsilon > 0. \)

For \( t \) sufficiently large, i.e., \( k \) sufficiently small, say \( 0 < k \leq k^* \), we have from the system (18)–(19)

\[
-\frac{dc}{dk} = -\frac{\dot{c}}{k} = \frac{\psi(c)[f'(k) - \rho - \delta]}{-(1/c)[f(k) - (n + \delta)k] + 1} \leq \frac{\psi f'(k)}{1/2} \equiv af'(k), \tag{30}
\]
where $\psi$ is an upper bound for $\psi(c)$. This immediately rules out $k_\varepsilon > 0$ since then $-\frac{dc}{dk}$ would be bounded above by a constant. If $k_\varepsilon = 0$, we find by integrating from $k$ to $k^*$ for $0 < k < k^*$

$$c(k) \leq c(k^*) + af(k^*), \quad 0 < k \leq k^*, \quad (31)$$

contradicting $c(k) \to \infty$ as $k \to 0^+$.

To get the separating orbit, $\Gamma$, consider a curve, $C$, connecting $(k, c) = (1, 0)$ with $(k, c) = (0, 1)$ and intersecting the nullcline $\dot{k} = 0$ once (think of a circle). We can write $\bar{C} = C_I \cup C_{II} \cup \{(1, 0), (0, 1)\}$ where $C_I$ and $C_{II}$ consists of the points through which pass orbits of class I and II, respectively. $C_{II}$ must be an open and connected part of $C$. Since $C_I$ and $C_{II}$ are both non-empty, $C_I \cup \{(1, 0)\}$ must be closed. The separating orbit $\Gamma$ goes through the end point of $C_I$. $\Box$

A powerful and useful extension of Lemma 1A is the simpler separator condition stated in:

**Corollary 1A.** With the assumptions of (23) and (25), the sufficient conditions for existence of the separating orbit, $\Gamma(t)$, cf. Lemma 1A, is given by the restriction

$$\bar{\psi} = \sup_{c > 0} \psi(c) < \frac{b - (n + \delta)}{\bar{b}}(\rho + \delta), \quad (32)$$

where $\bar{\psi}$ is the upper bound of the intertemporal substitution elasticity $\psi(c)$ of $u(c)$ and where $\bar{b}$ is given in (25).

**Proof.** Since $f(k) \to \infty$ as $k \to \infty$, it follows from (25) and l'Hospital that $y(k)/k \to b$ as $k \to \infty$. Thus for any number $\varepsilon > 0$, there exists a number, $k_\varepsilon$, such that for $k > k_\varepsilon$, we have, cf. (26)

$$f(k)/k - (n + \delta) - \psi(c)\left[f'(k) - (\delta + \rho)\right]$$

$$\geq \frac{b - \varepsilon - (n + \delta) - \sup_{c > 0} \psi(c)\left[b + \varepsilon - (\delta + \rho)\right]}{k}$$

$$= \frac{b - (n + \delta) - \bar{\psi}\left[b - (\delta + \rho)\right] - \varepsilon(1 + \bar{\psi})}{} \equiv d. \quad (33)$$

By assumption (25) and (32), the sum of the first three terms of $d$ is positive. Thus, by choosing $\varepsilon > 0$ sufficiently small, also $d$ is positive. Thus, the requirements of Lemma 1A are satisfied, and hence the separating orbit $\Gamma$ exists. $\Box$

**Lemma 2A.** If any selected utility function $u(c)$ is assumed to satisfy

$$\forall c \geq c_0 \geq 0: \quad 0 \leq u(c) \leq Ac, \quad (34)$$

where $A$ is any positive constant, then the convergence of the integral $U = \int_0^\infty u[c(t)]e^{-\rho t}dt$ for a solution $[k(t), c(t)]$ to the system (18)–(19) is assured, if

$$\bar{\psi} = \sup_{c > 0} \psi(c) < \frac{\rho - n}{\bar{b}}(\delta + \rho), \quad (35)$$

where $\bar{b}$ is given by (25).

**Proof.** Let $\varepsilon > 0$. Choose $k_\varepsilon$ such that $f'(k) < b + \varepsilon$ for $k \geq k_\varepsilon$. Choose $t_\varepsilon$ such that $k(t) > k_\varepsilon$ for $t > t_\varepsilon$. Then from (19), we find

$$\forall t \geq t_\varepsilon: \quad \dot{c} < c \sup_{c > 0} \psi(c)\left[b + \varepsilon - (\delta + \rho)\right] \equiv \alpha c. \quad (36)$$

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It follows from (36) that \( c(t) \leq c(t_\varepsilon)e^{\alpha(t-t_\varepsilon)} \), and hence, cf. (34)
\[
 u[c(t)]e^{-(\rho-n)t} \leq Ac(t)e^{-(\rho-n)t} \leq Ac(t_\varepsilon)e^{-\alpha t_\varepsilon}e^{-(\rho-n)-\alpha t_\varepsilon}. 
\] (37)

Thus the convergence of the integral \( U \) is assured, if \( \alpha < \rho - n \), which by (36) says
\[
\overline{\psi} = \sup_{c>0} \psi(c) < \frac{\rho - n}{b + \varepsilon - (\delta + \rho)}.
\] (38)

With \( \varepsilon > 0 \) chosen sufficiently small, the requirement (38) can be satisfied by the condition (35).

**Remark A.** The condition (35) is stronger that (32) of Corollary 1A, since by assumption (25), we have
\[
\frac{\rho - n}{b - (\rho + \delta)} < \frac{b - (n + \delta)}{b - (\rho + \delta)}. 
\] (39)

In short, the existence of separating orbit \( \Gamma \) is assured by \( \overline{\psi} < 1 \), but \( \overline{\psi} < 1 \) does not itself ensure convergence of \( U \). However, for isoelastic \( u(c) \) with \( \forall c, \psi(c) = \psi \) (constant), it can be verified that the convergence of \( U \) is in fact also ensured by the existence condition of the separating orbit, (32). Indeed, with constant intertemporal elasticity of substitution, the separating orbit in figure 6.2 is the optimal solution \([k^*(t), c^*(t)]\), satisfying the transversality condition; see hereto Gandolfo (1996, p. 390).

It remains to be seen how (35) may be relaxed for general non-isoelastic \( u(c) \) in Ramsey problems.

With regard to Ramsey (optimal) saving, it has been incumbent on us to obtain sufficient conditions – applicable to a general GNP-function, \( f(k) \) and a general utility function \( u(c) \) – that ensure persistent per capita growth. The condition (20) is analogous to with a low \( \rho \) taking over the role of a large \( s \). But (20) is not always enough to ensure persistent growth, as (21) is also needed. However, if \( u(c) \) always has \( \psi(c) \leq 1, \forall c > 0 \), then (21) is automatically satisfied (Hall 1988) estimated that \( \psi \) is much below unity, \( 0.1 < \psi < 0.4 \), irrespective of the size of \( \rho > 0 \). If \( \psi(c) > 1 \), then \( \rho \) must be large enough to satisfy (21).

Given now the existence of the separator \( \Gamma(t) \) and hence persistent growth, see Figure 1.2. The actual selection of the optimal path in region I was discussed above. For optimal saving and persistent growth, the factor shares also behave as in CES with \( \sigma > 1 \).

## 1 Numerical examples

In this section, we provide some numerical examples to illustrate the theory above. These examples are provided for two types of production functions, namely for extended Cobb-Douglas and CES functions. For the utility functions, \( u(c) \), a common practice is to use the class of isoelastic CRRA utility function. Rather than the parametrizations in von Thünen, it is convenient to have the risk parameter \( \rho_r(c) \) always located along the posetive real axis. Such convenient CRRA parametric form of \( u(c) \) is, see Barro and Sala-i-Martin (1995, p. 141), Solow (2000, p. 114) :
\[
u(c) = \frac{e^{1-\theta} - 1}{1 - \theta}; \quad \theta > 0; \quad \rho_r(c) = \theta; \quad \psi(c) = 1/\theta
\] (40)
For CRRA (40), the two conditions, (20), (21), for the existence of both endogenous growth and optimal solutions [a finite integral of V, (9), and the separator, (22)] can be summarized as:

$$b - \delta > \rho > (b - \delta)(1 - \theta) + n\theta$$  

(41)

The long-run (asymptotic) saving rate ($s^*$) is given by

$$s^* = 1 - x^*/(z^* = b) = 1 - \frac{\rho - (b - \delta)(1 - \theta) - n\theta}{b\theta}$$  

(42)

### 1.1 Baseline CES

Consider then the CES production function $Y = F(L, K)$ and the associated average and marginal products of capital:

$$Y = F(L, K) = \gamma \left[(1 - a)L^{\frac{\sigma-1}{\sigma}} + aK^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}; 0 < a < 1, \sigma > 0$$

$$AP_L(k) = Y/L = y = f(k) = \gamma \left[(1 - a) + ak^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

$$MP_{k}(k) = f'(k) = \gamma a \left[a + (1 - a)k^{\frac{-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} = \gamma \frac{\sigma}{\sigma-1} a [AP_{k}(k)]^\frac{\sigma-1}{\sigma}$$  

(43)

In the CES case (43), the dynamic system (18-19) becomes

$$\dot{k} = \gamma \left[(1 - a) + ak^{\frac{(\sigma-1)}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} - c - (n + \delta)k$$  

(46)

$$\dot{c} = (c/\theta)\left[\gamma a \left[a + (1 - a)k^{\frac{-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} - \delta - \rho\right]$$  

(47)

$$\dot{k} = 0 \Leftrightarrow c = \gamma \left[(1 - a) + ak^{\frac{-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} - (n + \delta)k.$$  

(48)

The transformed system then is as follows:

$$\dot{z} = -z[1 - \gamma \frac{\sigma-1}{\sigma} a z^{\frac{1}{\sigma}}][z - x - (n + \delta)]$$  

(49)

$$= -z[1 - (z/A)^{-\Psi}][z - x - (n + \delta)]$$

$$\dot{x} = x[z - x(1 - \frac{1}{\theta})(z/A)^{\frac{-1}{\sigma}} + n + \delta - \frac{\delta + \rho}{\theta}]$$  

(50)

$$= x[z - x(1 - \frac{1}{\theta}(z/A)^{-\Psi}) + n + \delta - \frac{\delta + \rho}{\theta}]$$  

(51)

$$\neq x[z - x(1 - \frac{1}{\theta}(z/A)^{-(\Psi+1)}) + n + \delta - \frac{\delta + \rho}{\theta}]$$  

Barro, (4.66)  

(52)

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1If they exist, the steady-state values of capital-labor ratios and per capita consumptions in optimal one-sector growth models are critical points of, (46), (47):

$$[\dot{c} = 0 \Leftrightarrow k(t) = \kappa] \Leftrightarrow \{f'(\kappa) = \rho + \delta\}; \quad MP_k(\kappa) = \rho + \delta$$  

(44)

$$[\dot{k} = 0 \Leftrightarrow c(t) = c(\kappa)] \Leftrightarrow \{c(\kappa) = f'(\kappa) - (n + \delta)\kappa\}$$  

(45)

The steady-state values, (44), (45), for an extensive set of CD and CES parameter cases are exhibited in Table 1. The steady state values of Table 1 are as is wellknown the saddle points depicted in Figure 1.1

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with isoclines and steady states

\begin{align*}
\dot{z} &= 0 \iff x = \gamma a \frac{z}{c} \quad \text{or} \quad x = z - (n + \delta) \quad (53) \\
\dot{x} &= 0 \iff x = z(1 - \frac{1}{\theta} \gamma a \frac{z}{c}) - (n + \delta) + \frac{\delta + \rho}{\theta} \quad (54) \\
z^* &= \gamma a \frac{z^*}{c} \quad (= A \text{ Barro, (4.65), } B = \gamma, \text{ no letter : } b) \quad (55) \\
x^* &= z^*(1 - \frac{1}{\theta}) - (n + \delta) + \frac{\delta + \rho}{\theta} \quad (= \varphi \text{ Barrop. } 231) \quad (56) \\
&= (z^* - \delta)(1 - \frac{1}{\theta}) + \frac{\rho}{\theta} - n \quad (= \varphi \text{ Barrop. } 231)
\end{align*}

Consider parameters of values \( \gamma = 1.6, \ a = 0.5, \ \sigma = 1.5, \ \delta = 0.03, \ \rho = 0.12, \ n = 0.01, \ \text{and } \theta = 5. \) For this CES-baseline parameter set, the asymptotic values are \( x^* = 0.15, \ \gamma^* = 0.20, \ \text{and } \delta^* = 0.25. \) Figures 3 and 6 below illustrate the CES-baseline set. Figure 3 shows the phase portrait for the system in the \( k, c \)-space (see also Barro and Sala-i-Martin 2004, fig. 4.3 and Gandolfo 1997, fig. 22.3) and Figure 4 shows the same system in the transformed space.
Figure 5 illustrates the time paths for capital accumulation and saving, showing how over-saving soon wedges the capital stock above its optimal path (separator) and capital starts to accumulate very fast as almost all output is devoted to saving. On the other hand, under-saving dilutes the capital stock to zero in some 30 – 40 years. Figure 6 for consumption thus illustrates that under-saving collapses consumption but, on the other hand, over-saving causes minor deviations from the optimal path in the short run. In the long run, however, over-saving starts to bite; the rightmost panel of Figure 6 shows that such consumption which lies 0.001 – 0.005% below its optimal level initially, causes it to fall some 50% below the optimum in hundred years. Nevertheless, the interesting conclusion is that there exist paths in the neighborhood of the separator which, in terms of short-run consumption, generate very similar outcomes than the optimal path.

1.2 Sensitivity analysis

Consider now variations in the CES-baseline parameter set above. Several types of sensitivity analysis are possible, including variations of one parameter at a time as well as simultaneous variations of several parameters. In this paper, we provide two new parameter sets, each of which generates exactly the same asymptotic outcome $x^* = 0.15$, $z^* = 0.20$, and $s^* = 0.25$ as the CES-baseline. In both parameter sets, only two parameters are varied. In the first set, denoted
as low − θ − high − ρ, we decrease the value of θ from 5 to 2 and increase the value of ρ from 0.12 to 0.15. In the second set (low − σ − high − γ), we decrease the value of σ from 1.5 to 1.2 and increase the value of γ from 1.60 to 12.80.

Figure 7: The time paths for the economic growth rates.

Figure 7, illustrating the optimal economic growth rates for each set, shows that considerable differences exist as the CES-baseline generates low and stable growth rates while low − σ − high − γ generates excessive and unstable ones and those for low − θ − high − ρ lie between these two extremes. Figure 8 shows the saving rates with analogous differences: for low − σ − high − γ the saving rates are high and decreasing but for CES-baseline low and increasing while low − θ − high − ρ again generates the intermediate values. The time paths of consumption and capital naturally respond to these findings. For low − σ − high − γ, for example, excessive capital accumulation takes place initially. Later, capital accumulation levels off but it exceeds that in the CES-baseline and low − θ − high − ρ even in the long run.

Figure 8: The time paths for the saving rates

To summarize, one can say that, in spite of asymptotic similarity, different parameter sets can produce very different temporal outcomes. Therefore, depending upon the values of the parameters, the endogenous growth model discussed here is able to explain several historical patterns of capital accumulation, saving and consumption.
2 Final Comments and Conclusion

References


A Extended Cobb-Douglas

Consider the extended Cobb-Douglas production function \( Y = F(L, K) \) and the associated average and marginal products of capital:

\[
Y = F(L, K) = AK + BK^\alpha \cdot L^{1-\alpha}, \\
y = Y/L = AF_L(k) = f(k) = Ak + Bk^\alpha, \\
MP_K(k) = f'(k) = A + \alpha Bk^{\alpha-1},
\]

where \( A > 0; \ B > 0; \ 0 < \alpha < 1 \). The dynamics of the Ramsay system discussed here are

\[
\dot{k} = f(k) - c - (n + \delta)k \\
\dot{c} = (c/\theta)[f'(k) - \sigma - \rho]
\]

(A.2) (A.3)

In the extended Cobb-Douglas case this implies

\[
\dot{k} = Ak + Bk^\alpha - c - (n + \delta)k \\
\dot{c} = (c/\theta)[A + \alpha Bk^{\alpha-1} - \sigma - \rho]
\]

(A.4) (A.5)

The isocline for \( k \) is:

\[
\dot{k} = 0 \iff c = Ak + Bk^\alpha - (n + \delta)k
\]

(A.6)

Given the transformations \( z = f(k)/k \) and \( x = c/k \), we can present the dynamics of the transformed system as follows:

\[
\dot{z} = -(1 - \alpha)(z - A)(z - x - n - \delta), \\
\dot{x} = x[(x - \varphi) - \frac{\theta - \alpha}{\theta} \cdot (z - A)],
\]

(A.7) (A.8)

where \( \varphi = (A - \delta) \cdot (\theta - 1)/\theta + \rho/\theta - n \). The isoclines of the transformed system are:

\[
\dot{z} = 0 \iff x = z - n - \delta \\
\dot{x} = 0 \iff x = \varphi + (z - A) - \frac{\alpha}{\theta} \cdot \left(\frac{z}{A}\right)^{1-\varphi} - 1
\]

(A.9) (A.10)

The steady state values for the transformed system are

\[
z^* = A \\
x^* = \varphi
\]

(A.11) (A.12)

To compare CES and Cobb-Douglas results, consider Cobb-Douglas parameters \( A = 0.20, \ B = 1.5, \) and \( \alpha = 0.5 \) and keep other parameters as in the CES-baseline (\( \delta = 0.03, \ \rho = 0.12, \ n = 0.01, \ \) and \( \theta = 5 \)). This parameter set generates exactly the same asymptotic values \( x^* = 0.15, \ z^* = 0.20, \) and \( s^* = 0.25 \) as the CES-baseline. Furthermore, Figure A shows that the Cobb-Douglas phase portrait is practically identical to its CES counterpart in Figure 3, and the time paths for capital and consumption in Figure A closely resemble those in Figures 5 and 6, indicating that, for suitable parameter sets, one can generate identical results for these two production functions.
Figure A: The phase portrait for the original space. Cobb-Douglas production function.

Figure A: The time paths for capital accumulation and consumption. Cobb-Douglas production function.